6. Tangents and Normals

Some Important Theorems

THEOREM 1 *Prove that the equation of a tangent to a curve* $y = f(x)$ *at a point* $P(x_1, y_1)$ *is given by* $\frac{y - y_1}{x - x_1} = \left(\frac{dy}{dx}\right)_{(x_1, y_1)}$,

where $\left(\frac{dy}{dx}\right)_{(x_1, y_1)}$ *denotes the value of* $\frac{dy}{dx}$ *at* $x = x_1$ *and* $y = y_1$,

and the equation of the normal at $P(x_1, y_1)$ *is given by*

$$\frac{y - y_1}{x - x_1} = \frac{-1}{\left(\frac{dy}{dx}\right)_{(x_1, y_1)}}.$$

PROOF We know that the slope of the tangent to the curve $y = f(x)$ at a point $P(x_1, y_1)$ is $\left(\frac{dy}{dx}\right)_{(x_1, y_1)}$.

Thus, the tangent to the given curve at a point P is a line passing through (x_1, y_1) with a slope of $\left(\frac{dy}{dx}\right)_{(x_1, y_1)}$.

So, the equation of the tangent at P is $\frac{y - y_1}{x - x_1} = \left(\frac{dy}{dx}\right)_{(x_1, y_1)}$.

Again, the normal to the given curve at P is a line perpendicular to the tangent at P.

So, the slope of the normal is $\frac{-1}{\left(\frac{dy}{dx}\right)_{(x_1, y_1)}}$.

Also, the normal at P passes through (x_1, y_1).

So, the equation of the normal at P is $\frac{y - y_1}{x - x_1} = \frac{-1}{\left(\frac{dy}{dx}\right)_{(x_1, y_1)}}$.

THEOREM 2 *The tangent to a curve* $y = f(x)$ *at a point* $P(x_1, y_1)$ *is parallel to the x-axis if and only if* $\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 0$.

PROOF The tangent is parallel to the x-axis $\Leftrightarrow$ its slope is 0

$$\Leftrightarrow \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 0.$$

THEOREM 3 *The tangent to a curve* $y = f(x)$ *at a point* $P(x_1, y_1)$ *is parallel to the y-axis if and only if* $\left(\frac{dx}{dy}\right)_{(x_1, y_1)} = 0$.

PROOF The tangent is parallel to the y-axis

$$\Leftrightarrow \text{ its slope is } 90^\circ \Leftrightarrow \left(\frac{dy}{dx}\right)_{(x_1,\, y_1)} = \tan 90^\circ = \infty \Leftrightarrow \left(\frac{dx}{dy}\right)_{(x_1,\, y_1)} = 0.$$

REMARK If $x = f(t)$ and $y = g(t)$, where $f'(t) \neq 0$ then

(i) equation of the tangent at 't' is $\dfrac{y - g(t)}{x - f(t)} = \dfrac{g'(t)}{f'(t)}$

and, (ii) equation of the normal at 't' is $\dfrac{y - g(t)}{x - f(t)} = -\dfrac{f'(t)}{g'(t)}$.

SOLVED EXAMPLES

EXAMPLE 1 *Find the equations of the tangent and the normal to the curve* $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ *at the point* $(1,\ 3)$.

SOLUTION The equation of the given curve is $y = x^4 - 6x^3 + 13x^2 - 10x + 5$.

$$\therefore \quad \frac{dy}{dx} = 4x^3 - 18x^2 + 26x - 10.$$

$$\text{So, } \left(\frac{dy}{dx}\right)_{(1,\, 3)} = (4 \times 1^3 - 18 \times 1^2 + 26 \times 1 - 10) = 2.$$

$\therefore$ the required equation of the tangent is

$$\frac{y - 3}{x - 1} = 2 \text{ or } 2x - y + 1 = 0.$$

And, the required equation of the normal is

$$\frac{y - 3}{x - 1} = \frac{-1}{2} \text{ or } x + 2y - 7 = 0.$$

EXAMPLE 2 *Find the equations of the tangent and the normal to the curve* $y = x^2 + 4x + 1$ *at the point where* $x = 3$.

SOLUTION When $x = 3$, we have $y = (3^2 + 4 \times 3 + 1) = 22$.

So, the point of contact is $(3,\ 22)$.

$$\text{Now, } y = x^2 + 4x + 1 \Rightarrow \frac{dy}{dx} = 2x + 4 \Rightarrow \left(\frac{dy}{dx}\right)_{(3,\, 22)} = (2 \times 3 + 4) = 10.$$

$$\text{Equation of the tangent is } \frac{y - 22}{x - 30} = 10 \Rightarrow 10x - y - 278 = 0.$$

$$\text{And, equation of the normal is } \frac{y - 22}{x - 3} = \frac{-1}{10} \Rightarrow x + 10y - 223 = 0.$$

EXAMPLE 3 *Show that the equation of the tangent to the ellipse*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ at } (x_1,\ y_1) \text{ is } \frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1.$$

SOLUTION $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \Rightarrow \dfrac{2x}{a^2} + \dfrac{2y}{b^2} \cdot \dfrac{dy}{dx} = 0$ [on differentiating w.r.t. x]

$$\frac{dy}{dx} = \frac{-b^2x}{a^2y} \Rightarrow \left(\frac{dy}{dx}\right)_{(x_1,\, y_1)} = \frac{-b^2x_1}{a^2y_1}.$$

So, the equation of the tangent at (x_1, y_1) is

$$\frac{y - y_1}{x - x_1} = \frac{-b^2x_1}{a^2y_1} \Rightarrow b^2xx_1 + a^2yy_1 = b^2x_1^2 + a^2y_1^2.$$

On dividing throughout by a^2b^2, we get $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2}$,

i.e., $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$ $\left[\because\ (x_1, y_1) \text{ lies on } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1\right]$.

EXAMPLE 4 *Find the equation of the tangent to the curve* $y = \sqrt{5x - 3} - 2$, *which is parallel to the line* $4x - 2y + 3 = 0$. **[CBSE 2005, '13C]**

SOLUTION The given line is $4x - 2y + 3 = 0$ or $y = 2x + \frac{3}{2}$.

$\therefore$ slope of the given line $= 2$.

So, the slope of the tangent $= 2$.

Let the point of contact be (x_1, y_1).

Now, $y = \sqrt{5x - 3} - 2 \Rightarrow \frac{dy}{dx} = \frac{5}{2 \cdot \sqrt{5x - 3}}$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(x_1,\, y_1)} = \frac{5}{2 \cdot \sqrt{5x_1 - 3}}.$$

$\therefore \frac{5}{2 \cdot \sqrt{5x_1 - 3}} = 2 \Rightarrow \frac{25}{4(5x_1 - 3)} = 4$, i.e., $x_1 = \frac{73}{80}$.

$\therefore\ y_1 = \sqrt{5x_1 - 3} - 2 = \sqrt{\left(5 \times \frac{73}{80} - 3\right)} - 2 = \frac{-3}{4}$.

So, the point of contact is $\left(\frac{73}{80}, \frac{-3}{4}\right)$.

Hence, the required equation of the tangent is $\frac{\left(y + \frac{3}{4}\right)}{\left(x - \frac{73}{80}\right)} = 2$,

i.e., $80x - 40y - 103 = 0$.

EXAMPLE 5 *Find the equations of the normals to the curve* $3x^2 - y^2 = 8$, *parallel to the line* $x + 3y = 4$.

SOLUTION The given line is $x + 3y = 4$ or $y = -\frac{1}{3}x + \frac{4}{3}$.

$\therefore$ slope of the given line $= -\frac{1}{3}$

So, the slope of the required normal $= -\frac{1}{3}$ … (i)

Let the point of contact be (x_1, y_1).

Now, $3x^2 - y^2 = 8 \Rightarrow 6x - 2y \cdot \frac{dy}{dx} = 0$ [on differentiating w.r.t. x]

$$\Rightarrow \frac{dy}{dx} = \frac{3x}{y} \Rightarrow \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{3x_1}{y_1}.$$

$$\therefore \text{ slope of the normal} = \frac{-1}{\left(\frac{dy}{dx}\right)_{(x_1, y_1)}} = \frac{-y_1}{3x_1} \qquad \ldots \text{(ii)}$$

Thus, from (i) and (ii) we get $\frac{-y_1}{3x_1} = -\frac{1}{3} \Rightarrow y_1 = x_1.$

Also, since (x_1, y_1) lies on the given curve, we have

$3x_1^2 - y_1^2 = 8$ or $3x_1^2 - x_1^2 = 8$ $[\because \quad y_1 = x_1]$

or $x_1^2 = 4$ or $x_1 = \pm 2.$

$\therefore \quad y_1 = \pm 2$ $[\because \quad y_1 = x_1].$

Thus, the points of contact are $(2, 2)$ and $(-2, -2)$.

So, the equation of the required normal at $(2, 2)$ is $\frac{y-2}{x-2} = \frac{-1}{3}$,

i.e., $x + 3y - 8 = 0.$

The equation of the required normal at $(-2, -2)$ is $\frac{y+2}{x+2} = \frac{-1}{3}$,

i.e., $x + 3y + 8 = 0.$

EXAMPLE 6 *Prove that the curve* $\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2$ *touches the straight line* $\frac{x}{a} + \frac{y}{b} = 2$ *at the point* (a, b), *whatever be the value of* n. **[CBSE 2007C]**

SOLUTION The equation of the curve is $\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2.$

On differentiating w.r.t. x, we get $n \cdot \left(\frac{x}{a}\right)^{n-1} \cdot \frac{1}{a} + n \cdot \left(\frac{y}{b}\right)^{n-1} \cdot \frac{1}{b} \cdot \frac{dy}{dx} = 0$

$$\Rightarrow \frac{n x^{n-1}}{a^n} + \frac{n y^{n-1}}{b^n} \cdot \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{b^n x^{n-1}}{a^n y^{n-1}}.$$

$$\therefore \left(\frac{dy}{dx}\right)_{(a, b)} = \frac{-b^n a^{n-1}}{a^n b^{n-1}} = -\frac{b}{a}.$$

So, the equation of the tangent at (a, b) is $\frac{y-b}{x-a} = \frac{-b}{a}$,

i.e., $ay + bx = 2ab.$

$\therefore \quad \frac{x}{a} + \frac{y}{b} = 2$, which is independent of n.

EXAMPLE 7 *At what points will the tangent to the curve* $y = 2x^3 - 15x^2 + 36x - 21$ *be parallel to the x-axis? Also, find the equations of tangents to the curve at these points.* **[CBSE 2008]**

SOLUTION Let the required point be $P(x_1, y_1)$.

Then, $\left[\frac{dy}{dx}\right]_{(x_1, y_1)} = 0.$

Now, $y = 2x^3 - 15x^2 + 36x - 21$

$\Rightarrow \quad \frac{dy}{dx} = 6x^2 - 30x + 36 = 6(x^2 - 5x + 6)$

$\Rightarrow \quad \left[\frac{dy}{dx}\right]_{(x_1, y_1)} = 6(x_1^2 - 5x_1 + 6).$

$\therefore \quad \left[\frac{dy}{dx}\right]_{(x_1, y_1)} = 0 \quad \Rightarrow \quad 6(x_1^2 - 5x_1 + 6) = 0$

$\Rightarrow \quad x_1^2 - 5x_1 + 6 = 0$

$\Rightarrow \quad (x_1 - 2)(x_1 - 3) = 0 \quad \Rightarrow \quad x_1 = 2 \text{ or } x_1 = 3.$

Since $P(x_1, y_1)$ lies on the given curve, we have

$y_1 = 2x_1^3 - 15x_1^2 + 36x_1 - 21.$

$\therefore \quad x_1 = 2 \quad \Rightarrow \quad y_1 = (2 \times 8 - 15 \times 4 + 36 \times 2 - 21) = 7.$

$x_1 = 3 \quad \Rightarrow \quad y_1 = (2 \times 27 - 15 \times 9 + 36 \times 3 - 21) = 6.$

So, the required points are $P(2, 7)$ and $P'(3, 6)$.

The equations of tangents at these points are $y = 7$ and $y = 6$ respectively.

EXAMPLE 8 *Prove that all points of the curve* $y^2 = 4a\left[x + a \sin\frac{x}{a}\right]$ *at which the tangent is parallel to the axis of x, lie on a parabola.*

SOLUTION Let the required point be (x_1, y_1).

Then, we must have $\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 0.$

Now, $y^2 = 4a\left[x + a \sin\frac{x}{a}\right] \Rightarrow 2y \cdot \frac{dy}{dx} = 4a\left[1 + \cos\frac{x}{a}\right]$

$\Rightarrow \quad \frac{dy}{dx} = \frac{2a\left[1 + \cos\frac{x}{a}\right]}{y} \Rightarrow \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{2a\left[1 + \cos\frac{x_1}{a}\right]}{y_1}.$

$\therefore \quad \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 0 \Rightarrow \frac{2a\left[1 + \cos\frac{x_1}{a}\right]}{y_1} = 0 \Rightarrow \left[1 + \cos\frac{x_1}{a}\right] = 0$

$\Rightarrow \quad \cos\frac{x_1}{a} = -1 \Rightarrow \sin\frac{x_1}{a} = \sqrt{1 - \cos^2\frac{x_1}{a}} = 0.$

But, (x_1, y_1) lies on the given curve.

$\therefore \quad y_1^2 = 4a\left[x_1 + a \sin\frac{x_1}{a}\right] \Rightarrow y_1^2 = 4ax_1 \qquad \left[\because \sin\frac{x_1}{a} = 0\right].$

This shows that (x_1, y_1) lies on the parabola $y^2 = 4ax$.

EXAMPLE 9 *Tangents are drawn from the origin to the curve* $y = \sin x$. *Prove that their points of contact lie on the curve* $x^2y^2 = (x^2 - y^2)$.

SOLUTION Let the point of contact be (x_1, y_1).

Now, $y = \sin x \Rightarrow \frac{dy}{dx} = \cos x \Rightarrow \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \cos x_1$.

$\therefore$ the equation of the tangent at (x_1, y_1) is $\frac{y - y_1}{x - x_1} = \cos x_1$.

Since the tangent passes through the origin, we have $\frac{-y_1}{-x_1} = \cos x_1$,

i.e., $\frac{y_1}{x_1} = \cos x_1$ … (i)

But, (x_1, y_1) lies on the curve $y = \sin x$.

$\therefore \quad y_1 = \sin x_1$ … (ii)

Squaring (i) and (ii) and adding, we get

$$\frac{y_1^2}{x_1^2} + y_1^2 = 1 \Rightarrow y_1^2 + x_1^2 y_1^2 = x_1^2$$

$$\Rightarrow x_1^2 y_1^2 = (x_1^2 - y_1^2).$$

This shows that (x_1, y_1) lies on the curve $x^2y^2 = (x^2 - y^2)$.

EXAMPLE 10 *Determine the points on the curve* $2y = (3 - x^2)$ *at which the tangent is parallel to the line* $x + y = 0$.

SOLUTION Let the required point be (x_1, y_1).

Then, slope of the tangent at $(x_1 \ \ y_1)$ = slope of the given line.

$\therefore \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = -1 \qquad [\because \ x + y = 0 \Rightarrow y = -x]$.

Now, $2y = (3 - x^2) \Rightarrow 2 \cdot \frac{dy}{dx} = -2x \Rightarrow \frac{dy}{dx} = -x \Rightarrow \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = -x_1$.

$\therefore \quad -x_1 = -1 \Rightarrow x_1 = 1$.

Since (x_1, y_1) lies on the curve $2y = (3 - x^2)$, we have $2y_1 = (3 - x_1^2)$.

When $x_1 = 1$, we have $y_1 = \left(\frac{3 - 1^2}{2}\right) = 1$.

Hence, the required point is (1, 1).

EXAMPLE 11 *Find the points on the curve* $4x^2 + 9y^2 = 1$, *where the tangents are perpendicular to the line* $2y + x = 0$.

SOLUTION The equation of the given line is $y = -\frac{1}{2}x$.

$\therefore$ slope of this line $= -\frac{1}{2}$ … (i)

Let the required point be (x_1, y_1).

Now, $4x^2 + 9y^2 = 1 \Rightarrow 8x + 18y \cdot \frac{dy}{dx} = 0$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{-4x}{9y}\right) \Rightarrow \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{-4x_1}{9y_1}.$$

$\therefore$ slope of the tangent at $(x_1, y_1) = \frac{-4x_1}{9y_1}$ … (ii)

From (i) and (ii), we have $\frac{-4x_1}{9y_1} \times \left(-\frac{1}{2}\right) = -1$ or $y_1 = -\frac{2}{9}x_1$ … (iii)

Since (x_1, y_1) lies on the curve $4x^2 + 9y^2 = 1$, we have

$$4x_1^2 + 9y_1^2 = 1 \Rightarrow 4x_1^2 + 9 \times \frac{4}{81}x_1^2 = 1 \quad \text{[using (iii)]}.$$

$\therefore \quad x_1^2 = \frac{9}{40}$ or $x_1 = \pm\frac{3}{2\sqrt{10}}$.

So, $y_1 = \pm\frac{2}{9} \times \frac{3}{2\sqrt{10}} = \pm\frac{1}{3\sqrt{10}}$.

Hence, the required points are

$$\left(\frac{3}{2\sqrt{10}}, \frac{1}{3\sqrt{10}}\right) \text{ and } \left(\frac{-3}{2\sqrt{10}}, \frac{-1}{3\sqrt{10}}\right).$$

EXAMPLE 12 *Find the coordinates of the points on the curve $y = x^2 + 3x + 4$, the tangents at which pass through the origin.*

SOLUTION Let the required point be (x_1, y_1).

Now, $y = x^2 + 3x + 4 \Rightarrow \frac{dy}{dx} = 2x + 3 \Rightarrow \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = (2x_1 + 3)$.

So, the equation of the tangent at (x_1, y_1) is $\frac{y - y_1}{x - x_1} = (2x_1 + 3)$.

Since the tangent passes through the origin, we have

$$\frac{-y_1}{-x_1} = (2x_1 + 3)$$

$$\Rightarrow \quad y_1 = (2x_1^2 + 3x_1) \quad \text{… (i)}$$

But, (x_1, y_1) lies on the given curve. So, $y_1 = x_1^2 + 3x_1 + 4$ … (ii)

From (i) and (ii), we get

$$2x_1^2 + 3x_1 = x_1^2 + 3x_1 + 4 \text{ or } x_1^2 = 4 \text{ or } x_1 = \pm 2.$$

Now, $x_1 = 2 \Rightarrow y_1 = (2^2 + 3 \times 2 + 4) = 14$.

And, $x_1 = -2 \Rightarrow y_1 = [(-2)^2 + 3 \times (-2) + 4] = 2$.

Hence, the required points are $(2, 14)$ and $(-2, 2)$.

EXAMPLE 13 *If the straight line $x\cos\alpha + y\sin\alpha = p$ touches the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, prove that $p^2 = a^2\cos^2\alpha + b^2\sin^2\alpha$.*

SOLUTION Let the point of contact be (x_1, y_1).

Then, the equation of the tangent at (x_1, y_1) is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$ ...(i)

This is identical with $x \cos\alpha + y \sin\alpha = p$ ... (ii)

Comparing coefficients, we have $\frac{(x_1/a^2)}{\cos\alpha} = \frac{(y_1/b^2)}{\sin\alpha} = \frac{1}{p}$

$$\Rightarrow \quad \frac{(x_1/a)}{a\cos\alpha} = \frac{(y_1/b)}{b\sin\alpha} = \frac{1}{p}.$$

$$\therefore \quad \frac{x_1}{a} = \frac{a\cos\alpha}{p} \text{ and } \frac{y_1}{b} = \frac{b\sin\alpha}{p}.$$

Squaring and adding, we get $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = \frac{a^2\cos^2\alpha + b^2\sin^2\alpha}{p^2}$

$$\Rightarrow \quad \frac{a^2\cos^2\alpha + b^2\sin^2\alpha}{p^2} = 1 \quad \left[\because \ \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1\right]$$

$$\Rightarrow \quad a^2\cos^2\alpha + b^2\sin^2\alpha = p^2.$$

EXAMPLE 14 *If $x\cos\alpha + y\sin\alpha = p$ touches the curve $x^m y^n = a^{m+n}$, prove that*

$$p^{m+n} \cdot m^m \cdot n^n = (m+n)^{m+n} \cos^m\alpha \sin^n\alpha.$$

SOLUTION Let the point of contact be (x_1, y_1).

The equation of the given curve is $x^m y^n = a^{m+n}$.

This, on differentiation, gives

$$mx^{m-1}y^n + x^m \cdot ny^{n-1} \cdot \frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{-my}{nx}.$$

$$\therefore \quad \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{-my_1}{nx_1}.$$

So, the equation of the tangent at (x_1, y_1) is $\frac{y - y_1}{x - x_1} = \frac{-my_1}{nx_1}$,

i.e., $my_1x + nx_1y = (m+n)\, x_1y_1$ ... (i)

This is identical with $x\cos\alpha + y\sin\alpha = p$ ... (ii)

Comparing coefficients, we get $\frac{my_1}{\cos\alpha} = \frac{nx_1}{\sin\alpha} = \frac{(m+n)x_1y_1}{p}$.

$$\therefore \quad x_1 = \frac{pm}{(m+n)\cos\alpha} \text{ and } y_1 = \frac{pn}{(m+n)\sin\alpha}.$$

Since (x_1, y_1) lies on the given curve, $x_1^m y_1^n = a^{m+n}$.

$$\therefore \quad \left\{\frac{pm}{(m+n)\cos\alpha}\right\}^m \cdot \left\{\frac{pn}{(m+n)\sin\alpha}\right\}^n = a^{m+n}$$

or $\quad p^{m+n} \cdot m^m \cdot n^n = (m+n)^{m+n} \cdot a^{m+n} \cos^m\alpha \sin^n\alpha.$

EXAMPLE 15 *Find the equation of the normal to the curve* $y = 2\sin^2 3x$ *at* $x = \frac{\pi}{6}$.

SOLUTION When $x = \frac{\pi}{6}$, we have $y = 2\sin^2\left(\frac{3\pi}{6}\right) = 2$.

So, the point of contact is $\left(\frac{\pi}{6}, 2\right)$.

Now, $y = 2\sin^2 3x \Rightarrow \frac{dy}{dx} = (4\sin 3x) \times 3 \times (\cos 3x)$

$\Rightarrow \frac{dy}{dx} = 12\sin 3x \cos 3x = 6\sin 6x.$

$\therefore \left(\frac{dy}{dx}\right)$ at $\left(x = \frac{\pi}{6}\right) = 6\sin\left(6 \times \frac{\pi}{6}\right) = 6\sin \pi = 0.$

So, the tangent is parallel to the x-axis.

Thus, the normal is parallel to the y-axis and passes through the point $\left(\frac{\pi}{6}, 2\right)$.

So, the equation of the normal is $x = \frac{\pi}{6}$.

EXAMPLE 16 *Find the equations of the tangent and the normal to the curve* $y(x-2)(x-3) - x + 7 = 0$ *at the point where it cuts the x-axis.*

[CBSE 2010]

SOLUTION The curve cuts the x-axis, where $y = 0$.

Putting $y = 0$ in the given equation, we get $x = 7$.

$\therefore$ the point of contact is $(7, 0)$.

Now, $y(x-2)(x-3) - x + 7 = 0 \Rightarrow y(x^2 - 5x + 6) - x + 7 = 0$

$\Rightarrow (x^2 - 5x + 6) \cdot \frac{dy}{dx} + y(2x - 5) - 1 = 0 \Rightarrow \frac{dy}{dx} = \frac{1 + 5y - 2xy}{x^2 - 5x + 6}$.

$\therefore \left(\frac{dy}{dx}\right)_{(7, 0)} = \left(\frac{1 + 5 \times 0 - 2 \times 7 \times 0}{49 - 35 + 6}\right) = \frac{1}{20}$.

So, the equation of the tangent is $\frac{y - 0}{x - 7} = \frac{1}{20}$ or $x - 20y - 7 = 0$.

Also, the equation of the normal is $\frac{y - 0}{x - 7} = -20$ or $20x + y - 140 = 0$.

EXAMPLE 17 *Show that* $\frac{x}{a} + \frac{y}{b} = 1$ *touches the curve* $y = be^{-x/a}$ *at the point where the curve crosses the axis of y.*

[CBSE 2005C, '07C]

SOLUTION The equation of the curve is $y = be^{-x/a}$.

It crosses the y-axis at the point where $x = 0$.

Putting $x = 0$ in the equation of the curve, we get $y = b$.

So, the point of contact is $(0, b)$.

Now, $y = be^{-x/a} \Rightarrow \frac{dy}{dx} = \frac{-be^{-x/a}}{a}$.

$$\therefore \quad \left(\frac{dy}{dx}\right)_{(0,\,b)} = -\frac{b}{a}.$$

So, the equation of the tangent is

$$\frac{y-b}{x-0} = \frac{-b}{a} \quad \Rightarrow \quad bx + ay = ab \quad \Rightarrow \quad \frac{x}{a} + \frac{y}{b} = 1.$$

Hence, $\frac{x}{a} + \frac{y}{b} = 1$ touches the curve $y = be^{-x/a}$ at $(0, b)$.

EXAMPLE 18 *Find the equations of the tangent and the normal at the point 't' on the curve* $x = a\sin^3 t,\ y = b\cos^3 t$. **[CBSE 2009C, '14]**

SOLUTION We have $\frac{dx}{dt} = 3a\sin^2 t\cos t$ and $\frac{dy}{dt} = -3b\cos^2 t\sin t$.

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-3b\cos^2 t\sin t}{3a\sin^2 t\cos t} = \frac{-b\cos t}{a\sin t}.$$

$\therefore$ the equation of the tangent at the point 't' is

$$\frac{y - b\cos^3 t}{x - a\sin^3 t} = \frac{-b\cos t}{a\sin t}$$

$$\Rightarrow \quad ay\sin t + bx\cos t = ab\sin t\cos t$$

$$\Rightarrow \quad \frac{x}{a\sin t} + \frac{y}{b\cos t} = 1 \quad \text{[on dividing throughout by } ab\sin t\cos t\text{]}.$$

Also, the equation of the normal at the point 't' is

$$\frac{y - b\cos^3 t}{x - a\sin^3 t} = \frac{a\sin t}{b\cos t}$$

$$\Rightarrow \quad ax\sin t - by\cos t = (a^2\sin^4 t - b^2\cos^4 t).$$

EXAMPLE 19 *Find the equations of the tangent and normal to the curve* $x = a\sin 3t,\ y = \cos 2t$ *at* $t = \frac{\pi}{4}$. **[CBSE 2008]**

SOLUTION We have

$$x = \sin 3t \ \Rightarrow \ \frac{dx}{dt} = 3\cos 3t$$

and $y = \cos 2t \ \Rightarrow \ \frac{dy}{dt} = -2\sin 2t.$

$$\therefore \quad \frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)} = \frac{-2\sin 2t}{3\cos 3t}$$

$$\Rightarrow \quad \left[\frac{dy}{dx}\right]_{t=\frac{\pi}{4}} = \frac{-2\sin\left(2\times\frac{\pi}{4}\right)}{3\cos\left(3\times\frac{\pi}{4}\right)} = \frac{-2\sin(\pi/2)}{-3\cos(\pi/4)} = \frac{2\sqrt{2}}{3}.$$

When $t = \frac{\pi}{4}$, we have $x = \sin\frac{3\pi}{4} = \sin\left(\pi - \frac{\pi}{4}\right) = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

and $\quad y = \cos\frac{2\pi}{4} = \cos\frac{\pi}{2} = 0.$

$\therefore$ point of contact is $P\left(\frac{1}{\sqrt{2}}, 0\right)$.

Equation of the tangent at the point P is given by

$$\frac{y-0}{x-\frac{1}{\sqrt{2}}} = \frac{2\sqrt{2}}{3}$$

$$\Rightarrow \quad \frac{\sqrt{2}y}{\sqrt{2}x-1} = \frac{2\sqrt{2}}{3}$$

$$\Rightarrow \quad 3\sqrt{2}y = 4x - 2\sqrt{2}$$

$$\Rightarrow \quad 2\sqrt{2}x - 3y - 2 = 0.$$

Equation of the normal at the point P is given by

$$\frac{y-0}{x-\frac{1}{\sqrt{2}}} = \frac{-3}{2\sqrt{2}} \quad \Rightarrow \quad \frac{\sqrt{2}y}{\sqrt{2}x-1} = \frac{-3}{2\sqrt{2}}$$

$$\Rightarrow \quad 4y = -3\sqrt{2}x + 3 \quad \Rightarrow \quad 3\sqrt{2}x + 4y - 3 = 0.$$

EXAMPLE 20 *For the curve $y = 4x^3 - 2x^5$, find all the points on the curve at which the tangent passes through the origin.* **[CBSE 2013C]**

SOLUTION The equation of the given curve is

$$y = 4x^3 - 2x^5. \qquad \ldots \text{(i)}$$

On differentiating (i) w.r.t. x, we get

$$\frac{dy}{dx} = 12x^2 - 10x^4.$$

Let the required point be $P(x, y)$.

The equation of tangent to the given curve at $P(x, y)$ is given by

$$Y - y = (12x^2 - 10x^4)(X - x). \qquad \ldots \text{(ii)}$$

If it passes through (0, 0) then, we have

$$y = (12x^2 - 10x^4)x$$

$$\Rightarrow \quad 4x^3 - 2x^5 = 12x^3 - 10x^5 \qquad \text{[using (i)]}$$

$$\Rightarrow \quad 8x^3 - 8x^5 = 0 \Rightarrow \quad 8x^3(1 - x^2) = 0$$

$$\Rightarrow \quad x = 0 \text{ or } x = 1 \text{ or } x = -1.$$

Putting $x = 0$ in (i), we get $y = 0$.

Putting $x = 1$ in (i), we get $y = 2$.

Putting $x = -1$ in (i), we get $y = -2$.

So, the required points are (0, 0), (1, 2) and (–1, –2).

EXERCISE 11H

1. Find the slope of the tangent to the curve

   (i) $y = (x^3 - x)$ at $x = 2$ (ii) $y = (2x^2 + 3\sin x)$ at $x = 0$

   (iii) $y = (\sin 2x + \cot x + 2)^2$ at $x = \frac{\pi}{2}$

Find the equations of the tangent and the normal to the given curve at the indicated point:

2. $y = x^3 - 2x + 7$ at (1, 6)
3. $y^2 = 4ax$ at $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$
4. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at $(a\cos 0,\ b\sin 0)$
5. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at $(a\sec 0,\ b\tan 0)$
6. $y = x^3$ at $P(1, 1)$ **[CBSE 2006C]**
7. $y^2 = 4ax$ at $(at^2, 2at)$
8. $y = \cot^2 x - 2\cot x + 2$ at $x = \frac{\pi}{4}$
9. $16x^2 + 9y^2 = 144$ at $(2, y_1)$, where $y_1 > 0$
10. $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ at the point where $x = 1$
11. Find the equation of the tangent to the curve $\sqrt{x} + \sqrt{y} = a$ at $\left(\frac{a^2}{4}, \frac{a^2}{4}\right)$.
12. Show that the equation of the tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at (x_1, y_1) is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$.
13. Find the equation of the tangent to the curve $y = (\sec^4 x - \tan^4 x)$ at $x = \frac{\pi}{3}$.
14. Find the equation of the normal to the curve $y = (\sin 2x + \cot x + 2)^2$ at $x = \frac{\pi}{2}$.
15. Show that the tangents to the curve $y = 2x^3 - 4$ at the points $x = 2$ and $x = -2$ are parallel. **[CBSE 2001C]**
16. Find the equation of the tangent to the curve $x^2 + 3y = 3$, which is parallel to the line $y - 4x + 5 = 0$. **[CBSE 2005]**
17. At what points on the curve $x^2 + y^2 - 2x - 4y + 1 = 0$, is the tangent parallel to the y-axis? **[CBSE 2000, '01C, '02C]**
18. Find the points on the curve $x^2 + y^2 - 2x - 3 = 0$ where the tangent is parallel to the x-axis. **[CBSE 2011]**
19. Prove that the tangents to the curve $y = x^2 - 5x + 6$ at the points (2, 0) and (3, 0) are at right angles.
20. Find the points on the curve $y = x^2 + 3x + 4$ at which the tangent passes through the origin.
21. Find the point on the curve $y = x^3 - 11x + 5$ at which the equation of tangent is $y = x - 11$. **[CBSE 2012]**
22. Find the equations of the tangents to the curve $2x^2 + 3y^2 = 14$, parallel to the line $x + 3y = 4$.

23. Find the equation of the tangent to the curve $x^2 + 2y = 8$, which is perpendicular to the line $x - 2y + 1 = 0$.
24. Find the point on the curve $y = 2x^2 - 6x - 4$ at which the tangent is parallel to the x-axis.
25. Find a point on the parabola $y = (x - 3)^2$, where the tangent is parallel to the chord joining the points (3, 0) and (4, 1). **[CBSE 2005C]**
26. Show that the curves $x = y^2$ and $xy = k$ cut at right angles if $8k^2 = 1$. **[CBSE 2005, '07C]**
27. Show that the curves $xy = a^2$ and $x^2 + y^2 = 2a^2$ touch each other. **[CBSE 2002]**
28. Show that the curves $x^3 - 3xy^2 + 2 = 0$ and $3x^2y - y^3 - 2 = 0$ cut orthogonally.
29. Find the equation of tangent to the curve $x = (\theta + \sin\theta)$, $y = (1 + \cos\theta)$ at $\theta = \frac{\pi}{4}$. **[CBSE 2006C]**
30. Find the equation of the tangent at $t = \frac{\pi}{4}$ for the curve $x = \sin 3t$, $y = \cos 2t$.

ANSWERS (EXERCISE 11H)

1. (i) 11 (ii) 3 (iii) –12 **2.** $x - y + 5 = 0,\ x + y - 7 = 0$

3. $m^2x - my + a = 0,\ m^2x + m^3y - 2am^2 - a = 0$

4. $bx\cos\theta + ay\sin\theta = ab,\ ax\sec\theta - by\,\text{cosec}\,\theta = (a^2 - b^2)$

5. $bx\sec 0 - ay\tan 0 = ab,\ by\cot 0 + ax\cos 0 = (a^2 + b^2)$

6. $y = 3x - 2,\ x + 3y = 4$ **7.** $x - ty + at^2 = 0,\ tx + y = at^3 + 2at$

8. $y = 1,\ x = \frac{\pi}{4}$ **9.** $8x + 3\sqrt{5}y - 36 = 0,\ 9\sqrt{5}x - 24y + 14\sqrt{5} = 0$

10. $2x - y + 1 = 0,\ x + 2y - 7 = 0$ **11.** $2(x + y) = a^2$

13. $3y - 48\sqrt{3}x + 16\sqrt{3}\pi - 21 = 0$ **14.** $24y - 2x + \pi - 96 = 0$

16. $4x - y + 13 = 0$ **17.** $(-1, 2)$ and $(3, 2)$

18. $(1, 2), (1, -2)$ **20.** $(-2, 2), (2, 14)$

21. $(2, -9)$ **22.** $x + 3y = 7,\ x + 3y = -7$

23. $2x + y - 6 = 0$ **24.** $\left(\frac{3}{2}, \frac{-17}{2}\right)$ **25.** $\left(\frac{7}{2}, \frac{1}{4}\right)$

29. $y = (1 - \sqrt{2})x + \frac{(\sqrt{2} - 1)\pi}{4} + 2$ **30.** $4x - 3\sqrt{2}y - 2\sqrt{2} = 0$

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 11H)

15. $x = 2 \Rightarrow y = (2 \times 8 - 4) = 12.$
$x = -2 \Rightarrow y = 2 \times (-8) - 4 = -20.$

$\frac{dy}{dx} = 6x^2 \Rightarrow \left(\frac{dy}{dx}\right)_{(2,\,12)} = 24$ and $\left(\frac{dy}{dx}\right)_{(x=-2)} = 24 \Rightarrow m_1 = m_2$.

16. $x^2 + 3y = 3 \Rightarrow 2x + 3\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-2x}{3}$.

$y - 4x + 5 = 0 \Rightarrow y = 4x + 5 \Rightarrow$ its slope $= 4$.

$\therefore$ the slope of the tangent $= 4$.

So, $\frac{-2x}{3} = 4 \Rightarrow x = -6$.

$x^2 + 3y = 3, x = -6 \Rightarrow y = -11$.

The equation of the tangent at $(-6, -11)$ with slope $= 4$ is $y + 11 = 4(x + 6)$.

17. $x^2 + y^2 - 2x - 4y + 1 = 0 \Rightarrow 2x\frac{dx}{dy} + 2y - 2\frac{dx}{dy} - 4 = 0 \Rightarrow \frac{dx}{dy} = \left(\frac{y-2}{1-x}\right)$.

Now, $\frac{dx}{dy} = 0 \Rightarrow \frac{y-2}{1-x} = 0 \Rightarrow y - 2 = 0 \Rightarrow y = 2$.

$x^2 + 4 - 2x - 8 + 1 = 0 \Rightarrow x^2 - 2x - 3 = 0 \Rightarrow (x - 3)(x + 1) = 0$

$\Rightarrow x = 3$ or $x = -1$.

$\therefore$ the required points are $(3, 2)$ and $(-1, 2)$.

18. $2a^2\frac{dy}{dx} = 3x^2 - 6ax$ and so, $\frac{dy}{dx} = 0 \Rightarrow 3x(x - 2a) = 0 \Rightarrow x = 0$ or $x = 2a$.

20. $y = x^2 + 3x + 4 \Rightarrow \frac{dy}{dx} = (2x + 3)$.

Let the required point be $P(a, b)$.

Then, $\left[\frac{dy}{dx}\right]_{(a,\,b)} = 2a + 3$.

$\therefore$ the tangent is $\frac{y-b}{x-a} = 2a + 3 \Rightarrow \frac{-b}{-a} = 2a + 3$ [$\because$ the tangent passes through $(0, 0)$].

$\therefore b = 2a^2 + 3a$ … (i)

Also, (a, b) lies on the given curve.

$\therefore b = a^2 + 3a + 4$ … (ii)

$\therefore 2a^2 + 3a = a^2 + 3a + 4 \Rightarrow a^2 = 4 \Rightarrow a = \pm 2$.

Now, $(a = 2 \Rightarrow b = 14)$ and $(a = -2 \Rightarrow b = 2)$.

21. The given curve is $y = x^3 - 11x + 5$. … (i)

Let (a, b) be the point on the given curve, the tangent at which is $y = x - 11$.

$\therefore$ slope of the tangent $= 1$.

Now, $y = x^3 - 11x + 5 \Rightarrow \frac{dy}{dx} = (3x^2 - 11) \Rightarrow \left(\frac{dy}{dx}\right)_{(a,\,b)} = (3a^2 - 11)$.

$3a^2 - 11 = 1 \Rightarrow 3a^2 = 12 \Rightarrow a^2 = 4 \Rightarrow a = \pm 2$.

Since the point (a, b) lies on the curve $y = x^3 - 11x + 5$, we have $b = a^3 - 11a + 5$.

Now, $a = 2 \Rightarrow b = (2^3 - 11 \times 2 + 5) = (13 - 22) = -9$.

And, $a = -2 \Rightarrow b = [(-2)^3 - 11 \times (-2) + 5] = (-8 + 27) = 19$.

Thus, the desired points can be $(2, -9)$ and $(-2, 19)$.

But, $(2, -9)$ satisfies (i) while $(-2, 19)$ does not satisfy it.

Hence, the required point is $(2, -9)$.

22. The given line is $x + 3y = 4 \Rightarrow y = -\frac{1}{3}x + \frac{4}{3}$.

Slope of tangent = slope of given line $= \frac{-1}{3}$.

Let the point of contact be (x_1, y_1).

Now, $2x^2 + 3y^2 = 14 \Rightarrow 4x + 6y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-4x}{6y} = \frac{-2x}{3y} \Rightarrow \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{-2x}{3y_1}$.

$\therefore \quad \frac{-2x_1}{3y_1} = \frac{-1}{3} \Rightarrow 2x_1 = y_1$

Also, (x_1, y_1) lies on the given curve.

$\therefore \quad 2x_1^2 + 3y_1^2 = 14 \Rightarrow 2x_1^2 + 3 \times 4x_1^2 = 14 \Rightarrow x_1^2 = 1 \Rightarrow x_1 = \pm 1.$

$(x_1 = 1 \Rightarrow y_1 = 2)$ and $(x_1 = -1 \Rightarrow y_1 = -2)$.

So, the points of contact are (1, 2) and (–1, –2).

The respective tangents are

$$\frac{y-2}{x-1} = \frac{-1}{3} \text{ and } \frac{y+2}{x+1} = \frac{-1}{3}$$

$\Rightarrow 3y - 6 = -x + 1$ and $3y + 6 = -x - 1 \Rightarrow 3y + x = 7$ and $3y + x = -7$.

25. Let the required point be $P(x_1, y_1)$.

Slope of the line joining $A(3, 0)$ and $B(4, 1)$ is $\frac{(1-0)}{(4-3)} = 1$. Now, $y = (x-3)^2$

$$\Rightarrow \frac{dy}{dx} = 2(x-3) \Rightarrow \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 2(x_1 - 3) \Rightarrow 2(x_1 - 3) = 1 \Rightarrow x_1 = \frac{7}{2}.$$

$$\therefore \quad y = (x-3)^2 \Rightarrow y_1 = (x_1 - 3)^2 = \left(\frac{7}{2} - 3\right)^2 = \frac{1}{4}.$$

Hence, the required point is $\left(\frac{7}{2}, \frac{1}{4}\right)$.

26. Let the point of intersection of the given curves be (x_1, y_1).

Then, $y^2 = x \Leftrightarrow 2y\frac{dy}{dx} = 1 \Leftrightarrow \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{1}{2y_1} \Leftrightarrow m_1 = \frac{1}{2y_1}$.

$$xy = k \Leftrightarrow x\frac{dy}{dx} + y = 0 \Leftrightarrow \frac{dy}{dx} = \frac{-y}{x} \Leftrightarrow \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = m_2 = \frac{-y_1}{x_1}.$$

$$\therefore \quad m_1 m_2 = -1 \Leftrightarrow \frac{1}{2y_1} \times \frac{-y_1}{x_1} = -1 \Leftrightarrow x_1 = \frac{1}{2}.$$

(x_1, y_1) lies on $y^2 = x \Leftrightarrow y_1^2 = x_1 \Leftrightarrow y_1^2 = \frac{1}{2}$.

(x_1, y_1) lies on $xy = k \Leftrightarrow x_1 y_1 = k \Leftrightarrow x_1^2 y_1^2 = k^2$

$$\Leftrightarrow k^2 = \left(\frac{1}{4} \times \frac{1}{2}\right) = \frac{1}{8}.$$

$$\Leftrightarrow 8k^2 = 1.$$

27. Solving the given equations, we get their points of intersection as (a, a) and $(-a, -a)$.

$$xy = a^2 \Rightarrow \frac{dy}{dx} = \frac{-y}{x} \Rightarrow \left(\frac{dy}{dx}\right)_{(a, a)} = -1 = m_1.$$

$$x^2 + y^2 = 2a^2 \Rightarrow \frac{dy}{dx} = \frac{-x}{y} \Rightarrow \left(\frac{dy}{dx}\right)_{(a, a)} = -1 = m_2.$$

$\therefore \quad \tan\phi = \left|\dfrac{m_1 - m_2}{1 + m_1 m_2}\right| = 0 \Rightarrow \phi = 0.$

So, the given curves touch each other.

28. $m_1 m_2 = -1.$

29. $x = (\theta + \sin\theta) \Rightarrow \dfrac{dx}{d\theta} = (1 + \cos\theta).$

$y = (1 + \cos\theta) \Rightarrow \dfrac{dy}{d\theta} = -\sin\theta.$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{(dy/d\theta)}{(dx/d\theta)} = \dfrac{-\sin\theta}{(1 + \cos\theta)}.$

$\left(\dfrac{dy}{dx}\right)$ at $\theta = \dfrac{\pi}{4} = \dfrac{-\sin(\pi/4)}{(1 + \cos\pi/4)} = \dfrac{-1}{\sqrt{2}} \cdot \dfrac{1}{(1 + 1/\sqrt{2})}$

$= \dfrac{-1}{(\sqrt{2} + 1)} \times \dfrac{(\sqrt{2} - 1)}{(\sqrt{2} - 1)} = (1 - \sqrt{2}).$

At $\theta = \dfrac{\pi}{4}$, we have $x = \left(\dfrac{\pi}{4} + \sin\dfrac{\pi}{4}\right) = \left(\dfrac{\pi}{4} + \dfrac{1}{\sqrt{2}}\right)$, and

$y = \left(1 + \cos\dfrac{\pi}{4}\right) = \left(1 + \dfrac{1}{\sqrt{2}}\right).$

Required equation of tangent is

$$\frac{y - \left(1 + \frac{1}{\sqrt{2}}\right)}{x - \left(\frac{\pi}{4} + \frac{1}{\sqrt{2}}\right)} = (1 - \sqrt{2}) \Rightarrow y = (1 - \sqrt{2})x + \frac{(\sqrt{2} - 1)\pi}{4} + 2.$$

OBJECTIVE QUESTIONS

Mark (✓) against the correct answer in each of the following:

1. If $y = 2^x$ then $\dfrac{dy}{dx} = ?$

(a) $x(2^{x-1})$ (b) $\dfrac{2^x}{(\log 2)}$ (c) $2^x(\log 2)$ (d) none of these

2. If $y = \log_{10} x$ then $\dfrac{dy}{dx} = ?$

(a) $\dfrac{1}{x}$ (b) $\dfrac{1}{x}(\log 10)$ (c) $\dfrac{1}{x(\log 10)}$ (d) none of these

3. If $y = e^{1/x}$ then $\dfrac{dy}{dx} = ?$

(a) $\dfrac{1}{x} \cdot e^{(1/x - 1)}$ (b) $\dfrac{-e^{1/x}}{x^2}$ (c) $e^{1/x} \log x$ (d) none of these

4. If $y = x^x$ then $\dfrac{dy}{dx} = ?$

(a) $x^x \log x$ (b) $x^x(1 + \log x)$ (c) $x(1 + \log x)$ (d) none of these

5. If $y = x^{\sin x}$ then $\frac{dy}{dx} = ?$

(a) $(\sin x) \cdot x^{(\sin x - 1)}$ (b) $(\sin x \cos x) \cdot x^{(\sin x - 1)}$

(c) $x^{\sin x}\left\{\frac{\sin x + x \log x \cdot \cos x}{x}\right\}$ (d) none of these

6. If $y = x^{\sqrt{x}}$ then $\frac{dy}{dx} = ?$

(a) $\sqrt{x} \cdot x^{(\sqrt{x} - 1)}$ (b) $\frac{x^{\sqrt{x}} \log x}{2\sqrt{x}}$ (c) $x^{\sqrt{x}}\left\{\frac{2 + \log x}{2\sqrt{x}}\right\}$ (d) none of these

7. If $y = e^{\sin \sqrt{x}}$ then $\frac{dy}{dx} = ?$

(a) $e^{\sin \sqrt{x}} \cdot \cos \sqrt{x}$ (b) $\frac{e^{\sin \sqrt{x}} \cos \sqrt{x}}{2\sqrt{x}}$ (c) $\frac{e^{\sin \sqrt{x}}}{2\sqrt{x}}$ (d) none of these

8. If $y = (\tan x)^{\cot x}$ then $\frac{dy}{dx} = ?$

(a) $\cot x \cdot (\tan x)^{\cot x - 1} \cdot \sec^2 x$ (b) $-(\tan x)^{\cot x} \cdot \operatorname{cosec}^2 x$

(c) $(\tan x)^{\cot x} \cdot \operatorname{cosec}^2 x (1 - \log \tan x)$ (d) none of these

9. If $y = (\sin x)^{\log x}$ then $\frac{dy}{dx} = ?$

(a) $(\log x) \cdot (\sin x)^{(\log x - 1)} \cdot \cos x$

(b) $(\sin x)^{\log x} \cdot \left\{\frac{x \log x + \log \sin x}{x}\right\}$

(c) $(\sin x)^{\log x} \cdot \left\{\frac{(x \log x) \cot x + \log \sin x}{x}\right\}$

(d) none of these

10. If $y = \sin (x^x)$ then $\frac{dy}{dx} = ?$

(a) $x^x \cos (x^x)$ (b) $x^x \cos x^x (1 + \log x)$

(c) $x^x \cos x^x \log x$ (d) none of these

11. If $y = \sqrt{x \sin x}$ then $\frac{dy}{dx} = ?$

(a) $\frac{(x \cos x + \sin x)}{2\sqrt{x \sin x}}$ (b) $\frac{1}{2}(x \cos x + \sin x) \cdot \sqrt{x \sin x}$

(c) $\frac{1}{2\sqrt{x \sin x}}$ (d) none of these

12. If $e^{x+y} = xy$ then $\frac{dy}{dx} = ?$

(a) $\frac{x(1-y)}{y(x-1)}$ (b) $\frac{y(1-x)}{x(y-1)}$ (c) $\frac{(x - xy)}{(xy - y)}$ (d) none of these

13. If $(x + y) = \sin(x + y)$ then $\frac{dy}{dx} = ?$

(a) –1 (b) 1 (c) $\frac{1 - \cos(x + y)}{\cos^2(x + y)}$ (d) none of these

14. If $\sqrt{x} + \sqrt{y} = \sqrt{a}$ then $\frac{dy}{dx} = ?$

(a) $\frac{-\sqrt{x}}{\sqrt{y}}$ (b) $-\frac{1}{2} \cdot \frac{\sqrt{y}}{\sqrt{x}}$ (c) $\frac{-\sqrt{y}}{\sqrt{x}}$ (d) none of these

15. If $x^y = y^x$ then $\frac{dy}{dx} = ?$

(a) $\frac{(y - x \log y)}{(x - y \log x)}$ (b) $\frac{y(y - x \log y)}{x(x - y \log x)}$ (c) $\frac{y(y + x \log y)}{x(x + y \log x)}$ (d) none of these

16. If $x^p y^q = (x + y)^{(p+q)}$ then $\frac{dy}{dx} = ?$

(a) $\frac{x}{y}$ (b) $\frac{y}{x}$ (c) $\frac{x^{p-1}}{y^{q-1}}$ (d) none of these

17. If $y = x^2 \sin \frac{1}{x}$ then $\frac{dy}{dx} = ?$

(a) $x \sin \frac{1}{x} - \cos \frac{1}{x}$ (b) $-\cos \frac{1}{x} + 2x \sin \frac{1}{x}$

(c) $-x \sin \frac{1}{x} + \cos \frac{1}{x}$ (d) none of these

18. If $y = \cos^2 x^3$ then $\frac{dy}{dx} = ?$

(a) $-3x^2 \sin(2x^3)$ (b) $-3x^2 \sin^2 x^3$ (c) $-3x^2 \cos^2(2x^3)$ (d) none of these

19. If $y = \log(x + \sqrt{x^2 + a^2})$ then $\frac{dy}{dx} = ?$

(a) $\frac{1}{2(x + \sqrt{x^2 + a^2})}$ (b) $\frac{-1}{\sqrt{x^2 + a^2}}$

(c) $\frac{1}{\sqrt{x^2 + a^2}}$ (d) none of these

20. If $y = \log\left(\frac{1 + \sqrt{x}}{1 - \sqrt{x}}\right)$ then $\frac{dy}{dx} = ?$

(a) $\frac{1}{\sqrt{x}(1 - x)}$ (b) $\frac{-1}{x(1 - \sqrt{x})^2}$ (c) $\frac{-\sqrt{x}}{2(1 - \sqrt{x})}$ (d) none of these

21. If $y = \log\left(\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}-x}\right)$ then $\frac{dy}{dx} = ?$

(a) $\frac{2}{\sqrt{1+x^2}}$ (b) $\frac{2\sqrt{1+x^2}}{x^2}$ (c) $\frac{-2}{\sqrt{1+x^2}}$ (d) none of these

22. If $y = \sqrt{\frac{1+\sin x}{1-\sin x}}$ then $\frac{dy}{dx} = ?$

(a) $\frac{1}{2}\sec^2\left(\frac{\pi}{4}-\frac{x}{2}\right)$ (b) $\frac{1}{2}\operatorname{cosec}^2\left(\frac{\pi}{4}-\frac{x}{2}\right)$

(c) $\frac{1}{2}\operatorname{cosec}\left(\frac{\pi}{4}-\frac{x}{2}\right)\cot\left(\frac{\pi}{4}-\frac{x}{2}\right)$ (d) none of these

23. If $y = \sqrt{\frac{\sec x - 1}{\sec x + 1}}$ then $\frac{dy}{dx} = ?$

(a) $\sec^2 x$ (b) $\frac{1}{2}\sec^2\frac{x}{2}$ (c) $\frac{-1}{2}\operatorname{cosec}^2\frac{x}{2}$ (d) none of these

24. If $y = \sqrt{\frac{1+\tan x}{1-\tan x}}$ then $\frac{dy}{dx} = ?$

(a) $\frac{1}{2}\sec^2 x \cdot \tan\left(x+\frac{\pi}{4}\right)$ (b) $\frac{\sec^2\left(x+\frac{\pi}{4}\right)}{2\sqrt{\tan\left(x+\frac{\pi}{4}\right)}}$

(c) $\frac{\sec^2\left(\frac{x}{4}\right)}{\sqrt{\tan\left(x+\frac{\pi}{4}\right)}}$ (d) none of these

25. If $y = \tan^{-1}\left(\frac{1-\cos x}{\sin x}\right)$ then $\frac{dy}{dx} = ?$

(a) 1 (b) –1 (c) $\frac{1}{2}$ (d) $\frac{-1}{2}$

26. If $y = \tan^{-1}\left\{\frac{\cos x + \sin x}{\cos x - \sin x}\right\}$ then $\frac{dy}{dx} = ?$

(a) 1 (b) –1 (c) $\frac{1}{2}$ (d) $\frac{-1}{2}$

27. If $y = \tan^{-1}\left\{\frac{\cos x}{1+\sin x}\right\}$ then $\frac{dy}{dx} = ?$

(a) $\frac{1}{2}$ (b) $\frac{-1}{2}$ (c) 1 (d) –1

28. If $y = \tan^{-1}\sqrt{\frac{1-\cos x}{1+\cos x}}$ then $\frac{dy}{dx} = ?$

(a) $\frac{-1}{2}$ (b) $\frac{1}{2}$ (c) $\frac{1}{(1+x^2)}$ (d) none of these

29. If $y = \tan^{-1}\left(\frac{a\cos x - b\sin x}{b\cos x + a\sin x}\right)$ then $\frac{dy}{dx} = ?$

(a) $\frac{a}{b}$ (b) $\frac{-b}{a}$ (c) 1 (d) –1

30. If $y = \sin^{-1}(3x - 4x^3)$ then $\frac{dy}{dx} = ?$

(a) $\frac{3}{\sqrt{1-x^2}}$ (b) $\frac{-4}{\sqrt{1-x^2}}$ (c) $\frac{3}{\sqrt{1+x^2}}$ (d) none of these

31. If $y = \cos^{-1}(4x^3 - 3x)$ then $\frac{dy}{dx} = ?$

(a) $\frac{3}{\sqrt{1-x^2}}$ (b) $\frac{-3}{\sqrt{1-x^2}}$ (c) $\frac{4}{\sqrt{1-x^2}}$ (d) $\frac{-4}{(3x^2-1)}$

32. If $y = \tan^{-1}\left(\frac{\sqrt{a}+\sqrt{x}}{1-\sqrt{ax}}\right)$ then $\frac{dy}{dx} = ?$

(a) $\frac{1}{(1+x)}$ (b) $\frac{1}{\sqrt{x}(1+x)}$ (c) $\frac{2}{\sqrt{x}(1+x)}$ (d) $\frac{1}{2\sqrt{x}(1+x)}$

33. If $y = \cos^{-1}\left(\frac{x^2-1}{x^2+1}\right)$ then $\frac{dy}{dx} = ?$

(a) $\frac{2}{(1+x^2)}$ (b) $\frac{-2}{(1+x^2)}$ (c) $\frac{2x}{(1+x^2)}$ (d) none of these

34. If $y = \tan^{-1}\left(\frac{1+x^2}{1-x^2}\right)$ then $\frac{dy}{dx} = ?$

(a) $\frac{2x}{(1+x^4)}$ (b) $\frac{-2x}{(1+x^4)}$ (c) $\frac{x}{(1+x^4)}$ (d) none of these

35. If $y = \cos^{-1}x^3$ then $\frac{dy}{dx} = ?$

(a) $\frac{-1}{(1+x)}$ (b) $\frac{2}{\sqrt{(1+x)}}$ (c) $\frac{-1}{2\sqrt{x}(1+x)}$ (d) none of these

36. If $y = \cos^{-1}x^3$ then $\frac{dy}{dx} = ?$

(a) $\frac{-1}{\sqrt{1-x^6}}$ (b) $\frac{-3x^2}{\sqrt{1-x^6}}$ (c) $\frac{-3}{x^2\sqrt{1-x^6}}$ (d) none of these

37. If $y = \tan^{-1}(\sec x + \tan x)$ then $\frac{dy}{dx} = ?$

(a) $\frac{1}{2}$ (b) $\frac{-1}{2}$ (c) 1 (d) none of these

38. If $y = \cot^{-1}\left(\frac{1-x}{1+x}\right)$ then $\frac{dy}{dx} = ?$

(a) $\frac{-1}{(1+x^2)}$ (b) $\frac{1}{(1+x^2)}$ (c) $\frac{1}{(1+x^2)^{3/2}}$ (d) none of these

39. If $y = \sqrt{\frac{1+x}{1-x}}$ then $\frac{dy}{dx} = ?$

(a) $\frac{2}{(1-x)^2}$ (b) $\frac{x}{(1-x)^{3/2}}$

(c) $\frac{1}{(1-x)^{3/2} \cdot (1+x)^{1/2}}$ (d) none of these

40. If $y = \sec^{-1}\left(\frac{x^2+1}{x^2-1}\right)$ then $\frac{dy}{dx} = ?$

(a) $\frac{-2}{(1+x^2)}$ (b) $\frac{2}{(1+x^2)}$ (c) $\frac{-1}{(1-x^2)}$ (d) none of these

41. If $y = \sec^{-1}\left(\frac{1}{2x^2-1}\right)$ then $\frac{dy}{dx} = ?$

(a) $\frac{-2}{(1+x^2)}$ (b) $\frac{-2}{(1-x^2)}$ (c) $\frac{-2}{\sqrt{1-x^2}}$ (d) none of these

42. If $y = \tan^{-1}\left\{\frac{\sqrt{1+x^2}-1}{x}\right\}$ then $\frac{dy}{dx} = ?$

(a) $\frac{1}{(1+x^2)}$ (b) $\frac{2}{(1+x^2)}$ (c) $\frac{1}{2(1+x^2)}$ (d) none of these

43. If $y = \sin^{-1}\left\{\frac{\sqrt{1+x}+\sqrt{1-x}}{2}\right\}$ then $\frac{dy}{dx} = ?$

(a) $\frac{-1}{2\sqrt{1-x^2}}$ (b) $\frac{1}{2\sqrt{1-x^2}}$ (c) $\frac{1}{2(1+x^2)}$ (d) none of these

44. If $x = at^2$, $y = 2at$ then $\frac{dy}{dx} = ?$

(a) $\frac{1}{t}$ (b) $\frac{-1}{t^2}$ (c) $\frac{-2}{t}$ (d) none of these

45. If $x = a\sec 0,\ y = b\tan\ 0$ then $\frac{dy}{dx} = ?$

(a) $\frac{b}{a}\sec\theta$ (b) $\frac{b}{a}\operatorname{cosec}\theta$ (c) $\frac{b}{a}\cot\theta$ (d) none of these

46. If $x = a\cos^2 0,\ y = b\sin^2 0$ then $\frac{dy}{dx} = ?$

(a) $\frac{-a}{b}$ (b) $\frac{a}{b}\cot\theta$ (c) $\frac{-b}{a}$ (d) none of these

47. If $x = a(\cos\theta + \theta\sin\theta)$ and $y = a(\sin\theta - \theta\cos\theta)$ then $\frac{dy}{dx} = ?$

(a) $\cot\theta$ (b) $\tan\theta$ (c) $a\cot\theta$ (d) $a\tan\theta$

48. If $y = x^{x^{x\ldots\infty}}$ then $\frac{dy}{dx} = ?$

(a) $\frac{y}{x(1-\log x)}$ (b) $\frac{y^2}{x(1-\log x)}$ (c) $\frac{y^2}{x(1-y\log x)}$ (d) none of these

49. If $y = \sqrt{x+\sqrt{x+\sqrt{x+}}}\ \ldots\infty$ then $\frac{dy}{dx} = ?$

(a) $\frac{1}{(2y-1)}$ (b) $\frac{1}{(y^2-1)}$ (c) $\frac{2y}{(y^2-1)}$ (d) none of these

50. If $y = \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x +}}}\ \ldots\infty$ then $\frac{dy}{dx} = ?$

(a) $\frac{\sin x}{(2y-1)}$ (b) $\frac{\cos x}{(y-1)}$ (c) $\frac{\cos x}{(2y-1)}$ (d) none of these

51. If $y = e^x + e^{x+\ldots\infty}$ then $\frac{dy}{dx} = ?$

(a) $\frac{1}{(1-y)}$ (b) $\frac{y}{(1-y)}$ (c) $\frac{y}{(y-1)}$ (d) none of these

52. The value of k for which $f(x) = \begin{cases} \frac{\sin 5x}{3x}, & \text{if } x \neq 0 \\ k, & \text{if } x = 0 \end{cases}$ is continuous at $x = 0$ is

(a) $\frac{1}{3}$ (b) 0 (c) $\frac{3}{5}$ (d) $\frac{5}{3}$

53. Let $f(x) = \begin{cases} x\sin\frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{when } x = 0. \end{cases}$

Then, which of the following is the true statement?

(a) $f(x)$ is not defined at $x = 0$ (b) $\lim_{x\to 0} f(x)$ does not exist

(c) $f(x)$ is continuous at $x = 0$ (d) $f(x)$ is discontinuous at $x = 0$

54. The value of k for which $f(x) = \begin{cases} \dfrac{3x + 4\tan x}{x}, & \text{when } x \neq 0 \\ k, & \text{when } x = 0 \end{cases}$

is continuous at $x = 0$, is

(a) 7 (b) 4 (c) 3 (d) none of these

55. Let $f(x) = x^{3/2}$. Then, $f'(0) = ?$

(a) $\frac{3}{2}$ (b) $\frac{1}{2}$ (c) does not exist (d) none of these

56. The function $f(x) = |x| \ \forall\, x \in R$ is

(a) continuous but not differentiable at $x = 0$

(b) differentiable but not continuous at $x = 0$

(c) neither continuous nor differentiable at $x = 0$

(d) none of these

57. The function $f(x) = \begin{cases} 1 + x, & \text{when } x \leq 2 \\ 5 - x, & \text{when } x > 2 \end{cases}$ is

(a) continuous as well as differentiable at $x = 2$

(b) continuous but not differentiable at $x = 2$

(c) differentiable but not continuous at $x = 2$

(d) none of these

58. If the function $f(x) = \begin{cases} kx + 5, & \text{when } x \leq 2 \\ x - 1, & \text{when } x > 2 \end{cases}$ is continuous at $x = 2$ then $k = ?$

(a) 2 (b) –2 (c) 3 (d) –3

59. If the function $f(x) = \begin{cases} \dfrac{1 - \cos 4x}{8x^2}, & x \neq 0 \\ k, & x = 0 \end{cases}$ is continuous at $x = 0$ then $k = ?$

(a) 1 (b) 2 (c) $\frac{1}{2}$ (d) $\frac{-1}{2}$

60. If the function $f(x) = \begin{cases} \dfrac{\sin^2 ax}{x^2}, & \text{when } x \neq 0 \\ k, & \text{when } x = 0 \end{cases}$ is continuous at $x = 0$ then $k = ?$

(a) a (b) a^2 (c) –2 (d) –4

61. If the function $f(x) = \begin{cases} \dfrac{k\cos x}{(\pi - 2x)}, & \text{when } x \neq \dfrac{\pi}{2} \\ 3, & \text{when } x = \dfrac{\pi}{2} \end{cases}$

be continuous at $x = \frac{\pi}{2}$, then the value of k is

(a) 3 (b) –3 (c) –5 (d) 6

62. At $x = 2$, $f(x) = [x]$ is
    (a) continuous but not differentiable
    (b) differentiable but not continuous
    (c) continuous as well as differentiable
    (d) none of these

63. Let $f(x) = \begin{cases} \dfrac{x^2 - 2x - 3}{x + 1^2}, & \text{when } x \neq -1 \\ k, & \text{when } x = -1. \end{cases}$
    If $f(x)$ is continuous at $x = -1$ then $k = ?$
    (a) 4 (b) -4 (c) -3 (d) 2

64. The function $f(x) = x^3 - 6x^2 + 15x - 12$ is
    (a) strictly decreasing on R
    (b) strictly increasing on R
    (c) increasing in $(-\infty, 2]$ and decreasing in $(2, \infty)$
    (d) none of these

65. The function $f(x) = 4 - 3x + 3x^2 - x^3$ is
    (a) decreasing on R (b) increasing on R
    (c) strictly decreasing on R (d) strictly increasing on R

66. The function $f(x) = 3x + \cos 3x$ is
    (a) increasing on R (b) decreasing on R
    (c) strictly increasing on R (d) strictly decreasing on R

67. The function $f(x) = x^3 - 6x^2 + 9x + 3$ is decreasing for
    (a) $1 < x < 3$ (b) $x > 1$ (c) $x < 1$ (d) $x < 1$ or $x > 3$

68. The function $f(x) = x^3 - 27x + 8$ is increasing when
    (a) $|x| < 3$ (b) $|x| > 3$ (c) $-3 < x < 3$ (d) none of these

69. $f(x) = \sin x$ is increasing in
    (a) $\left(\frac{\pi}{2}, \pi\right)$ (b) $\left(\pi, \frac{3\pi}{2}\right)$ (c) $(0, \pi)$ (d) $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

70. $f(x) = \dfrac{2x}{\log x}$ is increasing in
    (a) $(0, 1)$ (b) $(1, e)$ (c) (e, ∞) (d) $(-\infty, e)$

71. $f(x) = (\sin x - \cos x)$ is decreasing in
    (a) $\left(0, \frac{3\pi}{4}\right)$ (b) $\left(\frac{3\pi}{4}, \frac{7\pi}{4}\right)$ (c) $\left(\frac{7\pi}{4}, 2\pi\right)$ (d) none of these

72. $f(x) = \dfrac{x}{\sin x}$ is
    (a) increasing in $(0, 1)$
    (b) decreasing in $(0, 1)$

(c) increasing in $\left(0, \frac{1}{2}\right)$ and decreasing in $\left(\frac{1}{2}, 1\right)$

(d) none of these

73. $f(x) = x^x$ is decreasing in the interval

(a) $(0, e)$ (b) $\left(0, \frac{1}{e}\right)$ (c) $(0, 1)$ (d) none of these

74. $f(x) = x^2 e^{-x}$ is increasing in

(a) $(-2, 0)$ (b) $(0, 2)$ (c) $(2, \infty)$ (d) $(-\infty, \infty)$

75. $f(x) = \sin x - kx$ is decreasing for all $x \in R$, when

(a) $k < 1$ (b) $k \leq 1$ (c) $k > 1$ (d) $k \geq 1$

76. $f(x) = (x+1)^3 (x-3)^3$ is increasing in

(a) $(-\infty, 1)$ (b) $(-1, 3)$ (c) $(3, \infty)$ (d) $(1, \infty)$

77. $f(x) = [x(x-3)]^2$ is increasing in

(a) $(0, \infty)$ (b) $(-\infty, 0)$

(c) $(1, 3)$ (d) $\left(0, \frac{3}{2}\right) \cup (3, \infty)$

78. If $f(x) = kx^3 - 9x^2 + 9x + 3$ is increasing for every real number x, then

(a) $k > 3$ (b) $k \geq 3$ (c) $k < 3$ (d) $k \leq 3$

79. $f(x) = \frac{x}{(x^2+1)}$ is increasing in

(a) $(-1, 1)$ (b) $(-1, \infty)$

(c) $(-\infty, -1) \cup (1, \infty)$ (d) none of these

80. The least value of k for which $f(x) = x^2 + kx + 1$ is increasing on $(1, 2)$, is

(a) –2 (b) –1 (c) 1 (d) 2

81. $f(x) = |x|$ has

(a) minimum at $x = 0$

(b) maximum at $x = 0$

(c) neither a maximum nor a minimum at $x = 0$

(d) none of these

82. When x is positive, the minimum value of x^x is

(a) e^e (b) $e^{1/e}$ (c) $e^{-1/e}$ (d) $(1/e)$

83. The maximum value of $\left(\frac{\log x}{x}\right)$ is

(a) $\left(\frac{1}{e}\right)$ (b) $\frac{2}{e}$ (c) e (d) 1

84. $f(x) = \text{cosec } x$ in $(-\pi, 0)$ has a maxima at

(a) $x = 0$ (b) $x = \frac{-\pi}{4}$ (c) $x = \frac{-\pi}{3}$ (d) $x = \frac{-\pi}{2}$

85. If $x > 0$ and $xy = 1$, the minimum value of $(x + y)$ is

(a) –2 (b) 1 (c) 2 (d) none of these

86. The minimum value of $\left(x^2 + \frac{250}{x}\right)$ is

(a) 0 (b) 25 (c) 50 (d) 75

87. The minimum value of $f(x) = 3x^4 - 8x^3 - 48x + 25$ on $[0, 3]$ is

(a) 16 (b) 25 (c) –39 (d) none of these

88. The maximum value of $f(x) = (x-2)(x-3)^2$ is

(a) $\frac{7}{3}$ (b) 3 (c) $\frac{4}{27}$ (d) 0

89. The least value of $f(x) = (e^x + e^{-x})$ is

(a) –2 (b) 0 (c) 2 (d) none of these

ANSWERS (OBJECTIVE QUESTIONS)

1. (c)	**2.** (c)	**3.** (b)	**4.** (b)	**5.** (c)	**6.** (c)	**7.** (b)	**8** (c)	**9.** (c)	**10.** (b)
11. (a)	**12.** (b)	**13.** (a)	**14.** (c)	**15.** (b)	**16.** (b)	**17.** (b)	**18.** (a)	**19.** (c)	**20.** (a)
21. (a)	**22.** (b)	**23.** (b)	**24.** (b)	**25.** (c)	**26.** (a)	**27.** (b)	**28.** (b)	**29.** (d)	**30.** (a)
31. (b)	**32.** (d)	**33.** (b)	**34.** (a)	**35.** (c)	**36.** (b)	**37.** (a)	**38.** (b)	**39.** (c)	**40.** (a)
41. (c)	**42.** (c)	**43.** (a)	**44.** (a)	**45.** (b)	**46.** (c)	**47.** (b)	**48.** (c)	**49.** (a)	**50.** (c)
51. (b)	**52.** (d)	**53.** (c)	**54.** (a)	**55.** (c)	**56.** (a)	**57.** (b)	**58.** (b)	**59.** (c)	**60.** (b)
61. (d)	**62.** (d)	**63.** (b)	**64.** (b)	**65.** (a)	**66.** (a)	**67.** (a)	**68.** (b)	**69.** (d)	**70.** (c)
71. (b)	**72.** (a)	**73.** (b)	**74.** (b)	**75.** (c)	**76.** (d)	**77.** (d)	**78.** (a)	**79.** (a)	**80.** (a)
81. (a)	**82.** (c)	**83.** (a)	**84.** (d)	**85.** (c)	**86.** (d)	**87.** (c)	**88.** (c)	**89.** (c)	

HINTS TO THE GIVEN OBJECTIVE QUESTIONS

1. $\frac{dy}{dx} = 2^x(\log 2)$ 2. $y = \frac{\log x}{\log 10}$ 3. $\frac{dy}{dx} = \frac{e^{1/x}}{-x^2}$

4. $\log y = x \log x$. Now differentiate w.r.t. x.

5. $\log y = \sin x(\log x)$ 6. $\log y = \sqrt{x}(\log x)$ 7. $\log y = \sin \sqrt{x}$

8. $\log y = \cot x \cdot \log(\tan x)$ 9. $\log y = (\log x) \cdot (\log \sin x)$

10. Let $x^x = z$. Then, $y = \sin z$. Then, $\frac{dy}{dx} = \left(\frac{dy}{dz} \times \frac{dz}{dx}\right)$.

11. $y^2 = x\sin x \Rightarrow 2y\cdot\dfrac{dy}{dx} = (x\cos + \sin x).$

12. $(x + y) = \log x + \log y \Rightarrow 1 + \dfrac{dy}{dx} = \dfrac{1}{x} + \dfrac{1}{y}\cdot\dfrac{dy}{dx}.$

13. $(x + y) = \sin(x + y) \Rightarrow 1 + \dfrac{dy}{dx} = \cos(x + y)\cdot\left[1 + \dfrac{dy}{dx}\right].$

14. $\sqrt{x} + \sqrt{y} = \sqrt{a} \Rightarrow \dfrac{1}{2\sqrt{x}} + \dfrac{1}{2\sqrt{y}}\cdot\dfrac{dy}{dx} = 0.$

15. $x^y = y^x \Rightarrow y\log x = x\log y.$ Now, differentiate both sides w.r.t. x.

16. $p\log x + q\log y = (p + q)\log(x + y).$

17. $\dfrac{dy}{dx} = x^2\cdot\left(\cos\dfrac{1}{x}\right)\left(\dfrac{-1}{x^2}\right) + 2x\sin\dfrac{1}{x}.$

18. $y = (\cos x^3)^2 \Rightarrow \dfrac{dy}{dx} = 2(\cos x^3)(-\sin x^3)(3x^2) = -3x^2\sin(2x^3).$

19. $\dfrac{dy}{dx} = \dfrac{1}{(x + \sqrt{x^2 + a^2}}\cdot\left\{1 + \dfrac{1}{2\sqrt{x^2 + a^2}}\times 2x\right\} = \dfrac{1}{\sqrt{x^2 + a^2}}.$

20. $y = \log(1 + \sqrt{x}) - \log(1 - \sqrt{x})$

$\Rightarrow \quad \dfrac{dy}{dx} = \left\{\dfrac{1}{2\sqrt{x}(1 + \sqrt{x})} + \dfrac{1}{2\sqrt{x}(1 - \sqrt{x})}\right\} = \dfrac{1}{2\sqrt{x}}\cdot\dfrac{2}{(1 - x)} = \dfrac{1}{\sqrt{x}(1 - x)}.$

21. $y = \log(\sqrt{1 + x^2} + x) - \log(\sqrt{1 + x^2} - x).$ Now, differentiate.

22. $y = \left\{\dfrac{1 + \cos\left(\dfrac{\pi}{2} - x\right)}{1 - \cos\left(\dfrac{\pi}{2} - x\right)}\right\}^{\frac{1}{2}} = \left\{\dfrac{2\cos^2\left(\dfrac{\pi}{4} - \dfrac{x}{2}\right)}{2\sin^2\left(\dfrac{\pi}{4} - \dfrac{x}{2}\right)}\right\}^{\frac{1}{2}} = \cot\left(\dfrac{\pi}{4} - \dfrac{x}{2}\right).$

23. $y = \left(\dfrac{1 - \cos x}{1 + \cos x}\right)^{\frac{1}{2}} = \left\{\dfrac{2\sin^2(x/2)}{2\cos^2(x/2)}\right\}^{\frac{1}{2}} = \tan\dfrac{x}{2}.$

24. $y = \left\{\tan\left(x + \dfrac{\pi}{4}\right)\right\}^{\frac{1}{2}} \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{2}\left\{\tan\left(x + \dfrac{\pi}{4}\right)\right\}^{-\frac{1}{2}}\cdot\sec^2\left(x + \dfrac{\pi}{4}\right).$

25. $y = \tan^{-1}\left(\dfrac{1 - \cos x}{\sin x}\right) = \tan^{-1}\left\{\dfrac{2\sin^2(x/2)}{2\sin(x/2)\cos(x/2)}\right\} = \tan^{-1}\left\{\tan\dfrac{x}{2}\right\} = \dfrac{x}{2}.$

26. $y = \tan^{-1}\left\{\dfrac{\cos x + \sin x}{\cos x - \sin x}\right\} = \tan^{-1}\left\{\dfrac{1 + \tan x}{1 - \tan x}\right\} = \tan^{-1}\left\{\tan\left(\dfrac{\pi}{4} + x\right)\right\} = \left(\dfrac{\pi}{4} + x\right).$

27. $y = \tan^{-1}\left\{\dfrac{\sin\left(\dfrac{\pi}{2} - x\right)}{1 + \cos\left(\dfrac{\pi}{2} - x\right)}\right\} = \tan^{-1}\left\{\tan\left(\dfrac{\pi}{4} - \dfrac{x}{2}\right)\right\} = \left(\dfrac{\pi}{4} - \dfrac{x}{2}\right).$

28. $y = \tan^{-1}\left\{\dfrac{2\sin^2(x/2)}{2\cos^2(x/2)}\right\}^{\frac{1}{2}} = \tan^{-1}\left\{\tan\dfrac{x}{2}\right\} = \dfrac{x}{2}.$

29. $y = \tan^{-1}\left(\dfrac{\frac{a}{b} - \tan x}{1 + \frac{a}{b}\tan x}\right) = \tan^{-1}\left(\dfrac{\tan\theta - \tan x}{1 + \tan\theta\tan x}\right)$

$= \tan^{-1}\tan(\theta - x) = \theta - x = \left(\tan^{-1}\dfrac{a}{b} - x\right).$

$\therefore \quad \dfrac{dy}{dx} = -1.$

30. Putting $x = \sin\theta$, we get $y = \sin^{-1}(\sin 3\theta) = 3\theta = 3\sin^{-1}x.$

31. Putting $x = \cos\theta$, we get $y = \cos^{-1}(\cos 3\theta) = 3\theta = 3\cos^{-1}x.$

32. Put $\sqrt{a} = \tan\theta$ and $\sqrt{x} = \tan\phi$. Then,

$y = \tan^{1}\{\tan(\theta + \phi)\} = \theta + \phi = \tan^{-1}\sqrt{a} + \tan^{-1}\sqrt{x}.$

33. Putting $x = \cot\theta$, we get:

$y = \cos^{-1}\left(\dfrac{1 - \tan^2\theta}{1 + \tan^2\theta}\right) = \cos^{-1}(\cos 2\theta) = 2\theta = 2\cot^{-1}x.$

34. Putting $x^2 = \tan\theta$, we get:

$y = \tan^{-1}\left\{\tan\left(\dfrac{\pi}{4} + \theta\right)\right\} = \left(\dfrac{\pi}{4} + \theta\right) = \left(\dfrac{\pi}{4} + \tan^{-1}x^2\right).$

35. $y = \cot^{-1}\sqrt{x} \;\Rightarrow\; \dfrac{dy}{dx} = \dfrac{-1}{(1+x)}\cdot\dfrac{1}{2\sqrt{x}} = \dfrac{-1}{2\sqrt{x}(1+x)}.$

36. $y = \cos^{-1}x^3 \;\Rightarrow\; \dfrac{dy}{dx} = \dfrac{-3x^2}{\sqrt{1-x^6}}.$

37. $y = \tan^{-1}\left(\dfrac{1+\sin x}{\cos x}\right) = \tan^{-1}\left[\dfrac{\{\cos(x/2) + \sin(x/2)\}^2}{\cos^2(x/2) - \sin^2(x/2)}\right] = \tan^{-1}\left\{\dfrac{\cos(x/2) + \sin(x/2)}{\cos(x/2) - \sin(x/2)}\right\}$

$= \tan^{-1}\left\{\dfrac{1 + \tan(x/2)}{1 - \tan(x/2)}\right\} = \tan^{-1}\left\{\tan\left(\dfrac{\pi}{4} + \dfrac{x}{2}\right)\right\} = \left(\dfrac{\pi}{4} + \dfrac{x}{2}\right).$

38. Put $x = \tan\theta$. Then, $y = \cot^{-1}\cdot\tan\left(\dfrac{\pi}{4} - \theta\right) = \cot^{-1}\left[\cot\left\{\dfrac{\pi}{2} - \left(\dfrac{\pi}{4} - \theta\right)\right\}\right] = \left(\dfrac{\pi}{4} + \theta\right).$

$\therefore \quad y = \dfrac{\pi}{4} + \tan^{-1}x.$

39. $\log y = \dfrac{1}{2}\{\log(1+x) - \log(1-x)\} \;\Rightarrow\; \dfrac{1}{y}\cdot\dfrac{dy}{dx} = \dfrac{1}{2}\left\{\dfrac{1}{(1+x)} + \dfrac{1}{(1-x)}\right\}.$

40. Put $x = \cot\theta$. Then, $y = \sec^{-1}\left(\dfrac{1+\tan^2\theta}{1-\tan^2\theta}\right) = \sec^{-1}\left(\dfrac{1}{\cos 2\theta}\right) = \sec^{-1}(\sec 2\theta)$

$= 2\theta = 2\cot^{-1}x.$

41. Put $x = \cos\theta$. Then, $y = \sec^{-1}\left(\dfrac{1}{2\cos^2\theta - 1}\right) = \sec^{-1}(\sec 2\theta) = 2\theta = 2\cos^{-1}x$.

42. Put $x = \tan\theta$. Then, $y = \dfrac{1}{2}\tan^{-1}x$.

43. Put $x = \cos\theta$. Then, $y = \dfrac{\pi}{4} + \dfrac{1}{2}\cos^{-1}x$.

44. $\dfrac{dx}{dt} = 2at, \dfrac{dy}{dt} = 2a$. So, $\dfrac{dy}{dx} = \dfrac{(dy/dt)}{(dx/dt)} = \dfrac{1}{t}$.

45. $\dfrac{dx}{d\theta} = a\sec\theta\tan\theta, \dfrac{dy}{d\theta} = b\sec^2\theta \cdot \dfrac{dy}{dx} = \dfrac{(dy/d\theta)}{(dx/d\theta)} = \dfrac{b}{a}\operatorname{cosec}\theta$.

46. $\dfrac{dx}{d\theta} = -2a\cos\theta\sin\theta, \dfrac{dy}{d\theta} = 2b\sin\theta\cos\theta \Rightarrow \dfrac{dy}{dx} = \dfrac{(dy/d\theta)}{(dx/d\theta)} = \dfrac{-b}{a}$.

47. $\dfrac{dy}{dx} = \dfrac{(dy/d\theta)}{(dx/d\theta)}$.

48. $y = x^y \Rightarrow \log y = y\log x \Rightarrow \dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{y}{x} + (\log x)\dfrac{dy}{dx}$.

49. $y = \sqrt{x+y} \Rightarrow y^2 = x + y \Rightarrow 2y\dfrac{dy}{dx} = 1 + \dfrac{dy}{dx}$.

50. $y = \sqrt{\sin x + y} \Rightarrow y^2 = \sin x + y \Rightarrow 2y\dfrac{dy}{dx} = \cos x + \dfrac{dy}{dx}$.

51. $y = e^{x\,|\,y} \Rightarrow x + y = \log y \Rightarrow 1 + \dfrac{dy}{dx} = \dfrac{1}{y}\cdot\dfrac{dy}{dx}$.

52. For continuity at $x = 0$, we must have $\lim\limits_{x\to 0} f(x) = f(0)$.

$\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0}\dfrac{\sin 5x}{5x}\times\dfrac{5}{3} = \dfrac{5}{3}\lim\limits_{5x\to 0}\dfrac{\sin 5x}{5x} = \left(\dfrac{5}{3}\times 1\right) = \dfrac{5}{3}$.

$\therefore$ we must have, $f(0) = \dfrac{5}{3} \Leftrightarrow k = \dfrac{5}{3}$.

53. $f(0) = 0$.

$\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} x\sin\dfrac{1}{x} = 0 \times$ (a finite quantity) $= 0$.

$\therefore$ $f(x)$ is continuous at $x = 0$.

54. $f(0) = k$.

$\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0}\dfrac{3x + 4\tan x}{x} = \lim\limits_{x\to 0}\left\{3 + \dfrac{4\tan x}{x}\right\} = (3 + 4) = 7$.

$\therefore$ $f(x)$ is continuous at $x = 0 \Leftrightarrow f(0) = 7 \Leftrightarrow k = 7$.

55. $f'(0) = \lim\limits_{h\to 0}\dfrac{f(0-h) - f(0)}{-h} = \lim\limits_{h\to 0}\dfrac{(h^{3/2} - 0)}{-h} = \lim\limits_{h\to 0}(-h)^{1/2}$, which does not exist, since $(-h)^{1/2}$ is imaginary.

56. $f(0+0) = \lim\limits_{h\to 0}|0 + h| = \lim\limits_{h\to 0}|h| = 0$,

$f(0-0) = \lim\limits_{h\to 0}|0 - h| = \lim\limits_{h\to 0}|-h| = \lim\limits_{h\to 0}|h| = 0$ and $f(0) = 0$

$\therefore$ $f(x)$ is continuous at $x = 0$

$$Rf'(0) = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{f(h)-0}{h} = \lim_{h\to 0}\frac{|h|}{h} = \lim_{h\to 0}\frac{h}{h} = 1$$

$$Lf'(0) = \lim_{h\to 0}\frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0}\frac{f(-h)-0}{-h} = \lim_{h\to 0}\frac{|-h|}{-h} = \lim_{h\to 0}\frac{h}{-h} = -1.$$

$\therefore$ $Rf'(0) \neq Lf'(0)$, which shows that $f(x)$ is not differentiable at $x = 0$.

57. $f(2+0) = \lim_{h\to 0} f(2+h) = \lim_{h\to 0}\{5-(2+h)\} = 3$;

$f(2-0) = \lim_{h\to 0} f(2-h) = \lim_{h\to 0}\{1+(2-h)\} = 3$ and $f(2) = 3$.

$\therefore$ $f(x)$ is continuous at $x = 2$

$$Rf'(2) = \lim_{h\to 0}\frac{f(2+h)-f(2)}{h} = \lim_{h\to 0}\frac{(3-h)-3}{h} = \lim_{h\to 0}\frac{-h}{h} = -1.$$

$$Lf'(2) = \lim_{h\to 0}\frac{f(2-h)-f(2)}{-h} = \lim_{h\to 0}\frac{(3-h)-3}{-h} = \lim_{h\to 0}\frac{h}{h} = 1.$$

$\therefore$ $f(x)$ is not differentiable at $x = 2$.

58. $f(2) = \lim_{x\to 2} f(x)$.

$$\lim_{x\to 2^+} f(x) = \lim_{h\to 0} f(2+h) = \lim_{h\to 0}(2+h-1) = 1$$

$$\lim_{x\to 2^-} f(x) = \lim_{h\to 0} f(2-h) = \lim_{h\to 0}\{k(2-h)+5\} = 2k+5$$

$\therefore$ $2k + 5 = 1 \Rightarrow k = -2$.

Also, $f(2) = 2k + 5 = 1$. Hence, $k = -2$.

59. $$\lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0}\frac{1-\cos 4h}{8h^2} = \lim_{h\to 0}\frac{2\sin^2 2h}{8h^2}$$

$$= \frac{1}{2}\lim_{h\to 0}\left(\frac{\sin 2h}{2h}\right)^2 = \left(\frac{1}{2}\times 1^2\right) = \frac{1}{2}.$$

$$\lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0}\frac{1-\cos 4(-h)}{8(-h)^2} = \lim_{h\to 0}\frac{(1-\cos 4h)}{8h^2} = \frac{1}{2}.$$

$\therefore$ $\lim_{x\to 0^+} f(x) = \frac{1}{2}$

For continuity, we must have $f(0) = \frac{1}{2}$.

60. $$\lim_{x\to 0} f(x) = \lim_{x\to 0}\frac{\sin^2 ax}{a^2x^2}\times a^2 = a^2\cdot\lim_{ax\to 0}\left(\frac{\sin ax}{ax}\right)^2 = a^2\times 1^2 = a^2.$$

For continuity, we must have $f(0) = a^2$.

61. $$f\left(\frac{\pi}{2}-0\right) = \lim_{h\to 0}\frac{k\cos\left(\frac{\pi}{2}-h\right)}{\pi-2\left(\frac{\pi}{2}-h\right)} = \lim_{h\to 0}\frac{k\sin h}{2h} = \frac{k}{2}\lim_{h\to 0}\frac{\sin h}{h} = \left(\frac{k}{2}\times 1\right) = \frac{k}{2}.$$

$$f\left(\frac{\pi}{2}+0\right) = \lim_{h\to 0}\frac{k\cos\left(\frac{\pi}{2}+h\right)}{\pi-2\left(\frac{\pi}{2}+h\right)} = \lim_{h\to 0}\frac{-k\sin h}{-2h} = \frac{k}{2}\lim_{h\to 0}\frac{\sin h}{h} = \frac{k}{2}$$

$\therefore$ $\frac{k}{2} = 3 \Rightarrow k = 6$.

62. $f(2) = [2] = 2.$

$f(2+0) = \lim_{h\to 0} f(2+h) = \lim_{h\to 0} [2+h] = 2.$

$f(2-0) = \lim_{h\to 0} f(2-h) = \lim_{h\to 0} [2-h] = 1.$

$\therefore$ $f(x)$ is not continuous at $x = 2.$

$Rf'(2) = \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0} \frac{[2+h]-[2]}{h} = \lim_{h\to 0} \frac{(2-2)}{h} = 0.$

$Lf'(2) = \lim_{h\to 0} \frac{f(2-h)-f(2)}{-h} = \lim_{h\to 0} \frac{[2-h]-[2]}{-h} = \lim_{h\to 0} \frac{(1-2)}{-h} = \lim_{h\to 0} \frac{1}{h} = \infty$

$\therefore$ $f(x)$ is not differentiable at $x = 2.$

63. $\lim_{x\to -1} f(x) = \lim_{x\to -1} \frac{(x-3)(x+1)}{(x+1)} = \lim_{x\to -1} (x-3) = -4.$

For continuity, we must have, $f(-1) = -4.$

64. $f'(x) = 3x^2 - 12x + 15 = 3(x^2 - 4x + 5) = 3[(x-2)^2 + 1] > 0.$

$\Rightarrow$ $f'(x) > 0$ for all $x \in R$ $\Rightarrow$ $f(x)$ is strictly increasing on $R.$

65. $f'(x) = -3 + 6x - 3x^2 = -3(x^2 - 2x + 1) = -3(x-1)^2 \le 0.$

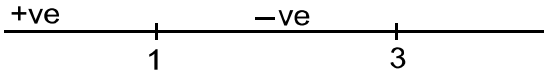
$\Rightarrow$ $f'(x) \le 0$ for all $x \in R$ $\Rightarrow$ $f(x)$ is decreasing on $R.$

66. $f'(x) = 3 - 3\sin 3x = 3(1 - \sin 3x) \ge 0$ since $-1 \le \sin 3x \le 1.$

$\therefore$ $f'(x) \ge 0$ for all $x \in R$ $\Rightarrow$ $f(x)$ is increasing on $R.$

67. $f'(x) = 3x^2 - 12 + 9 = 3(x^2 - 4x + 3) = 3(x-1)(x-3)$

$f'(x) = 0 \Rightarrow x = 1$ or $x = 3.$

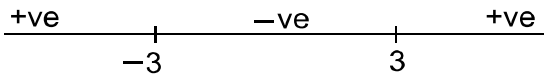


There are two factors in $f'(x)$, so we start with $+ve$ sign.

$\therefore$ $f(x)$ is decreasing for $1 < x < 3.$

68. $f'(x) = 3x^2 - 27 = 3(x^2 - 9) = 3(x+3)(x-3)$

$\therefore$ $f'(x) = 0 \Rightarrow x = -3$ or $x = 3.$



There are two factors in $f'(x)$, so we start with $+ve$ sign.

$\therefore$ $f(x)$ is increasing when $x < -3$ or $x > 3$, i.e., when $|x| > 3.$

69. $f'(x) = \cos x > 0$ in $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right) \Rightarrow f(x)$ is increasing in $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right).$

70. $f'(x) = \frac{(\log x)\cdot 2 - 2x\cdot\frac{1}{x}}{(\log x)^2} = \frac{2(\log x - 1)}{(\log x)^2}$

$\therefore$ $f'(x) > 0 \Leftrightarrow \log x - 1 > 0 \Leftrightarrow \log x > 1 \Leftrightarrow \log x > \log e \Leftrightarrow x > e.$

$\therefore$ $f(x)$ is increasing in $(e, \infty).$

71. $f'(x) = (\cos x + \sin x) = \sqrt{2}\left(\frac{1}{\sqrt{2}}\cos x + \frac{1}{\sqrt{2}}\sin x\right) = \sqrt{2}\sin\left(\frac{\pi}{4} + x\right)$

$\therefore$ $f'(x) < 0 \Rightarrow \sin\left(\frac{\pi}{4} + x\right) < 0 \Rightarrow \pi < \frac{\pi}{4} + x < 2\pi \Rightarrow \frac{3\pi}{4} < x < \frac{7\pi}{4}.$

$\therefore$ $f(x)$ is decreasing in $\left(\frac{3\pi}{4}, \frac{7\pi}{4}\right).$

72. $f'(x) = \dfrac{\sin x - x\cos x}{\sin^2 x} = \dfrac{\cos x(\tan x - x)}{\sin^2 x}$.

$0 < x < 1 \Rightarrow \tan x > x$ and $\cos x > 0 \Rightarrow \cos x(\tan x - x) > 0$

$\Rightarrow f'(x) > 0$

$\therefore$ $f(x)$ is increasing in (0, 1).

73. $f'(x) = x^x(1 + \log x)$

$f'(x) < 0 \Leftrightarrow (1 + \log x) < 0 \Rightarrow \log x < -1 = \log \dfrac{1}{e} \Rightarrow x > 0$ and $x < \dfrac{1}{e}$.

$\therefore$ $f(x)$ is decreasing in $\left(0, \dfrac{1}{e}\right)$.

74. $f'(x) = 2xe^{-x} - x^2e^{-x} = xe^{-x}(2 - x)$

$\therefore$ $f'(x) > 0 \Leftrightarrow x > 0$ and $(2 - x) > 0 \Leftrightarrow 0 < x < 2$.

$\therefore$ $f(x)$ is increasing in (0, 2).

75. $f'(x) = (\cos x - k)$ and therefore,

$f(x)$ is decreasing $\Leftrightarrow f'(x) < 0 \Rightarrow \cos x - k < 0$

$\Rightarrow \cos x < k \Rightarrow k > \cos x \Rightarrow k > 1$.

76. $f(x) = (x + 1)^3(x - 3)^3$

$\therefore$ $f'(x) = 3(x + 1)^3(x - 3)^2 + 3(x + 1)^2(x - 3)^3$

$= 3(x + 1)^2(x - 3)^2[(x + 1) + (x - 3)] = 3(x + 1)^2(x - 3)^2(x - 1)$

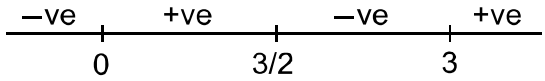
$\Rightarrow f'(x) > 0$ when $(x - 1) > 0$, i.e., when $x > 1$.

$\therefore$ $f(x)$ is increasing in $(1, \infty)$.

77. $f(x) = [x(x - 3)]^2 \Rightarrow f'(x) = 2x(x - 3)(2x - 3)$.

$\therefore$ $f'(x) = 0 \Rightarrow x = 0$ or $x = \dfrac{3}{2}$ or $x = 3$.

$f(x)$ is increasing when $0 < x < \dfrac{3}{2}$ or $x > 3$.



$\therefore$ $f(x)$ is increasing in $\left(0, \dfrac{3}{2}\right) \cup (3, \infty)$.

78. $f'(x) = 3kx^2 - 18x + 9 = 3(kx^2 - 6x + 3)$.

This is positive when $k > 0$ and $(36 - 12k) < 0 \Rightarrow k > 3$.

79. $f'(x) = \dfrac{(x^2 + 1)\cdot 1 - x \cdot 2x}{(x^2 + 1)^2} = \dfrac{(1 - x^2)}{(1 + x^2)^2}$.

$f'(x) > 0 \Leftrightarrow (1 - x^2) > 0 \Leftrightarrow x^2 < 1 \Leftrightarrow -1 < x < 1$.

$\therefore$ $f(x)$ is increasing in (–1, 1).

80. $f'(x) = (2x + k)$.

$1 < x < 2 \Rightarrow 2 < 2x < 4 \Rightarrow 2 + k < 2x + k < 4 + k \Rightarrow 2 + k < f'(x) < 4 + k$.

$f(x)$ is increasing $\Leftrightarrow (2x + k) \geq 0 \Leftrightarrow 2 + k \geq 0 \Leftrightarrow k \geq -2$.

$\therefore$ least value of k is –2.

81. $f(x) = |x| \geq 0$ for all $x \in R$.

The least value of $|x|$ is 0 at $x = 0$.

$\therefore$ $f(x) = |x|$ has minima at $x = 0$.

82. $f(x) = x^2 \Rightarrow f'(x) = x^x(1 + \log x)$ and $f''(x) = x^x\left[\frac{1}{x} + (1 + \log x)^2\right]$.

$f'(x) = 0 \Rightarrow 1 + \log x = 0 \Rightarrow \log x = -1 = \log\left(\frac{1}{e}\right) \Rightarrow x = \left(\frac{1}{e}\right)$.

$[f''(x)]_{x=(1/e)} = e\left(\frac{1}{e}\right)^{1/e} > 0.$

$\therefore \quad x = \frac{1}{e}$ is a point of minima.

Minimum value of x^x is $\left(\frac{1}{e}\right)^{1/e} = e^{-1/e}$.

83. $f(x) = \frac{\log x}{x} \Rightarrow f'(x) = \frac{x \cdot \frac{1}{x} - \log x \cdot 1}{x^2} = \frac{(1 - \log x)}{x^2}$

$$\Rightarrow f''(x) = \frac{x^2 \cdot \frac{1}{x} - (1 - \log x) \cdot 2x}{x^4} = \frac{(-3 + 2\log x)}{x^3}$$

$\therefore \quad f'(x) = 0 \Rightarrow 1 - \log x = 0 \Rightarrow \log x = 1 = \log e \Rightarrow x = e.$

$f''(e) = \left(\frac{-3 + 2}{e^3}\right) = \frac{-1}{e^3} < 0.$

$\therefore \quad x = e$ is a point of maxima.

Maximum value of $f(x)$ is $\frac{1}{e}(\log e) = \frac{1}{e}$.

84. $f(x) = \text{cosec } x \Rightarrow f'(x) = -\text{cosec } x \cot x$

$$\Rightarrow f''(x) = \text{cosec}^3 x + \text{cosec } x(\cot^2 x) = \text{cosec } x(\text{cosec}^2 x + \cot^2 x)$$

$$= \text{cosec } x(2\text{cosec}^2 x - 1)$$

$f'(x) = 0 \Rightarrow \cot x = 0 \Rightarrow x = \frac{-\pi}{2}$

$f''\left(\frac{-\pi}{2}\right) = \text{cosec}\left(\frac{-\pi}{2}\right)\left[2\text{cosec}^2\left(\frac{-\pi}{2}\right) - 1\right] = (-1)(2 - 1) = -1 < 0.$

$\therefore \quad x = \frac{-\pi}{2}$ is a point of maxima.

85. $xy = 1 \Rightarrow y = \frac{1}{x}$.

Let $S = x + y = x + \frac{1}{x}$. Then, $\frac{ds}{dx} = \left(1 - \frac{1}{x^2}\right) = \frac{(x^2 - 1)}{x^2}$ and $\frac{d^2s}{dx^2} = \frac{2}{x^3}$.

$\frac{ds}{dx} = 0 \Rightarrow x^2 - 1 = 0 \Rightarrow x = \pm 1.$

$\left.\frac{d^2s}{dx^2}\right]_{(x=-1)} = -2 < 0$ and $\left.\frac{d^2s}{dx^2}\right]_{(x=1)} = 2 > 0$

$\therefore \quad S$ is minimum at $x = 1$ and minimum value of $S = (1 + 1) = 2$.

86. Let $f(x) = \left(x^2 + \frac{250}{x}\right)$. Then, $f'(x) = \left(2x - \frac{250}{x^2}\right)$ and $f''(x) = \left(2 + \frac{500}{x^3}\right)$

$f'(x) = 0 \Rightarrow 2x^3 - 250 = 0 \Rightarrow x^3 = 125 \Rightarrow x = 5.$

$f''(5) = \left(2 + \frac{500}{125}\right) = 6 > 0$

$\therefore$ $f(x)$ is minimum at $x = 5$ and minimum value $= \left(25 + \frac{250}{5}\right) = 75.$

87. $f'(x) = 12x^3 - 24x^2 + 24x - 48 = 12(x - 2)(x^2 + 2)$

$f''(x) = 36x^2 - 48x + 24 = 12(3x^2 - 4x + 2).$

$f'(x) = 0 \Rightarrow x = 2$ and $f''(2) = 12(3 \times 4 - 4 \times 2 + 2) = 72 > 0$

$\therefore$ $x = 2$ is a point of minima.

Minimum value $= \min\{f(0), f(2), f(3)\}$

$= \min\{25, -39, 16\} = -39.$

88. $f(x) = (x - 2)(x - 3)^2 \Rightarrow f'(x) = (x - 3)(3x - 7)$ and $f''(x) = (6x - 16).$

$f'(x) = 0 \Rightarrow x = 3$ or $x = \frac{7}{3}$

$f''(3) = 2 > 0$ and $f''\left(\frac{7}{3}\right) = -2 < 0.$

$\therefore$ $x = \frac{7}{3}$ is a point of maxima.

Maximum value $= \left(\frac{7}{3} - 2\right)\left(\frac{7}{3} - 3\right)^2 = \frac{4}{27}.$

89. $f'(x) = e^x - e^{-x}$ and $f''(x) = e^x + e^{-x}.$

$f'(x) = 0 \Rightarrow e^x - e^{-x} = 0 \Rightarrow e^x = e^{-x} \Rightarrow e^{2x} = e^0 \Rightarrow x = 0.$

$f''(0) = e^0 + \frac{1}{e^0} = (1 + 1) = 2 > 0.$

$\therefore$ $f(x)$ is minimum at $x = 0$ and minimum value of $f(x)$ is 2.